The two-body relativistic bound state

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We start with the equations of motion of quantum electrodynamics

$$(i\partial\!\!\!/ - e A\!\!\!/ - m)\psi(x) = 0 \tag{1}$$
$$(\partial^2 + \mu^2)A^\mu(x) = e\overline{\psi}(x)\gamma^\mu\psi(x) \tag{2}$$

where we allow for a possible photon mass. In terms of Fourier transforms defined by

$$\psi(p) = (2\pi)^{-4}\int d^4x\, e^{-ip\cdot x}\, \psi(x) \tag{3}$$

and similarly for A_μ, these equations become

$$(p\!\!\!/ + m)\psi(p) = -e\int d^4q\, A\!\!\!/(q)\psi(p-q) \tag{4}$$
$$(p^2 - \mu^2)A^\mu(p) = -e\int d^4q\,\overline{\psi}(q)\gamma^\mu\psi(p+q) \tag{5}$$

Here we consider a particle-antiparticle system of two non-identical fundamental fermions ψ and χ with the same charge. Combining (4), and (5) for the χ species, and considering just the χ current, we have

$$\psi_\alpha(p) = -\left(\frac{e}{p\!\!\!/ + m}\right)_\alpha{}^\beta \int d^4q \frac{-e}{q^2-\mu^2}\int d^4q' (\overline{\chi}_0(q')\gamma^\mu\chi_0(q+q'))(\gamma_\mu\psi(p-q))_\beta \tag{6}$$

where we are using early Greek letters for Dirac spinor indices. We have approximated χ with its free-field value χ_0. Now apply $\overline{\chi}_0^\beta(-p_1)|\rangle$ to the right, where $|\rangle$ is the vacuum state. This gives

$$\psi_\alpha(p_2)\overline{\chi}_0^\beta(-p_1)|\rangle = \left(\frac{e^2}{p\!\!\!/_2 + m}\right)_\alpha{}^\gamma \int \frac{d^4q}{q^2-\mu^2} d^4q' (\overline{\chi}_0(q')\gamma^\mu\chi_0(q+q'))(\gamma_\mu\psi(p_2-q))_\gamma\overline{\chi}_0^\beta(-p_1)|\rangle \tag{7}$$

χ_0 can be anticommuted past ψ and then $\overline{\chi}_0$ to annihilate the vacuum. We then pick up the positive-energy part of the anticommutator function

$$\{\chi_{0\alpha}(p), \overline{\chi}_0^\beta(q)\} = (p\!\!\!/ - M)_\alpha{}^\beta\delta(p-q)\delta(p^2-M^2)\epsilon(p_0) \tag{8}$$

where M is the χ fermion mass. This gives

$$\psi_\alpha(p_2)\overline{\chi}_0^\beta(-p_1)|\rangle = e^2(p\!\!\!/_2 + m)^{-1}{}_\alpha{}^\gamma \int \frac{d^4q}{q^2-\mu^2}(\overline{\chi}_0(-q-p_1)\gamma^\mu)^\delta(\gamma_\mu\psi(p_2-q))_\gamma$$
$$(-p\!\!\!/_1 - M)_\delta{}^\beta\delta(p_1^2-M^2)\theta(p_1^0)|\rangle \tag{9}$$

Thus

$$\psi_\alpha(p_2)\overline{\chi}_0^\beta(-p_1)|\rangle = e^2\delta(p_1^2-M^2)\theta(p_1^0)(\gamma^\mu(p\!\!\!/_1 + M))_\delta{}^\beta((p\!\!\!/_2+m)^{-1}\gamma_\mu)_\alpha{}^\gamma \int\frac{d^4q}{q^2-\mu^2}\psi_\gamma(p_2-q)\overline{\chi}_0^\delta(-q-p_1)|\rangle \tag{10}$$

Write $p_2 = p - p_1$. Then

$$\psi_\alpha(p-p_1)\overline{\chi}_0^\beta(-p_1)|\rangle = e^2\delta(p_1^2-M^2)\theta(p_1^0)(\gamma^\mu(p\!\!\!/_1+M))_\delta{}^\beta((p\!\!\!/ - p\!\!\!/_1 + m)^{-1}\gamma_\mu)_\alpha{}^\gamma$$
$$\int\frac{d^4q}{(q-p_1)^2-\mu^2}\psi_\gamma(p-q)\overline{\chi}_0^\delta(-q)|\rangle \tag{11}$$

Multiplying through by $\not{p} - \not{p}_1 + m$, we have

$$(\not{p} - \not{p}_1 + m)_\alpha{}^\gamma \psi_\gamma(p - p_1)\overline{\chi}_0^\beta(-p_1)|\rangle = e^2\delta(p_1^2 - M^2)\theta(p_1^0) \int \frac{d^4q}{(q - p_1)^2 - \mu^2}(\gamma_\mu)_\alpha{}^\gamma \psi_\gamma(p - q)\overline{\chi}_0^\delta(-q)$$
$$(\gamma^\mu(\not{p}_1 + M))_\delta{}^\beta|\rangle \quad (12)$$

We then write

$$\psi_\alpha(p - p_1)\overline{\chi}_0^\beta(-p_1)|\rangle = g_\alpha{}^\beta(p, p_1)\delta(p_1^2 - M^2)\theta(p_1^0) \quad (13)$$

From which we find

$$(\not{p} - \not{p}_1 + m)g(p, p_1) = e^2 \int \frac{d^4q\theta(q_0)\delta(q^2 - M^2)}{(q - p_1)^2 - \mu^2}\gamma^\mu g(p, q)\gamma_\mu(\not{p}_1 + M) \quad (14)$$

where we have suppressed the Dirac spinor indices. This is similar, if not actually identical to Cowen-Greenberg equation 30. One of the differences is conventions: we must replace e^2 with $e^2/(2\pi)^3$ to compare. This factor arises because equation 8 here conventionally has an extra $(2\pi)^3$.

Here we attempt to solve the equation in the chiral representation, where we explicitly show the Dirac spinor as a direct sum of two-component SL(2,C) objects:

$$\psi_\alpha(p) = \begin{bmatrix} \psi_A(p) \\ \overline{\tilde{\psi}}^{A'}(-p) \end{bmatrix}; \qquad \text{and} \qquad \overline{\psi}^\beta(-p) = \begin{bmatrix} \tilde{\psi}^B(p) & \overline{\psi}_{B'}(-p) \end{bmatrix} \quad (15)$$

and similarly for χ_0. In the chiral representation, with Penrose conventions for SL(2,C) we have

$$\gamma^\mu = \sqrt{2}\begin{bmatrix} 0 & \sigma^\mu_{BA'} \\ \sigma^{\mu AB'} & 0 \end{bmatrix} \quad (16)$$

Since χ_0 satisfies the free Dirac equation $\overline{\chi}_0(-p_1)(-\not{p}_1 + M) = 0$, we can write

$$\overline{\chi}_0^\beta(-p) = \begin{bmatrix} \tilde{\chi}_0^B(p) & \frac{\sqrt{2}}{M}p_{AB'}\tilde{\chi}_0^A(p) \end{bmatrix} \quad (17)$$

Substituting into (12), we find just one independent equation for the β index, which is

$$[(\not{p} - \not{p}_1 + m)\psi(p - p_1)]_\alpha \tilde{\chi}_0^B(p_1)|\rangle = 2e^2\delta(p_1^2 - M^2)\theta(p_1^0) \int \frac{d^4q}{(q - p_1)^2 - \mu^2)}[\gamma_\mu\psi(p - q)]_\alpha$$
$$(\sigma^{\mu BC'}q_{CC'} + p_1^{BC'}\sigma^\mu_{CC'})\tilde{\chi}_0^C(q)|\rangle \quad (18)$$

We may now write out ψ in terms of two components as well. This gives

$$\begin{bmatrix} m\psi_A(p - p_1) + \sqrt{2}(p - p_1)_{AB'}\overline{\tilde{\psi}}^{B'}(-p + p_1) \\ \sqrt{2}(p - p_1)^{DA'}\psi_D(p - p_1) + m\overline{\tilde{\psi}}^{A'}(-p + p_1) \end{bmatrix} \tilde{\chi}_0^B(p_1)|\rangle = 2\sqrt{2}e^2\delta(p_1^2 - M^2)\theta(p_1^0) \int \frac{d^4q}{(q - p_1)^2 - \mu^2}$$
$$\begin{bmatrix} (\delta_A^B q_{CB'} - \epsilon_{AC}p_1^B{}_{B'})\overline{\tilde{\psi}}^{B'}(-p + q) \\ (\epsilon^{DB}q_C{}^{A'} + \delta_C^D p_1^{BA'})\psi_D(p - q) \end{bmatrix} \tilde{\chi}_0^C(q)|\rangle \quad (19)$$

Solving for $\overline{\tilde{\psi}}\tilde{\chi}_0|\rangle$ in the bottom equation, and resubstituting into the top equation, we get

$$((p - p_1)^2 - m^2)\psi_A(p - p_1)\tilde{\chi}'_{0B}(p_1)|\rangle = 4e^2 \int \frac{dM(q)}{(q - p_1)^2 - \mu^2}R_{AB}^{CD}(p, p_1, q)\psi_C(p - q)\tilde{\chi}'_{0D}(q)|\rangle -$$
$$8e^4 \int \frac{dM(q)}{(q - p_1)^2 - \mu^2}\frac{dM(q')}{(q - q')^2 - \mu^2}S_{AB}^{CD}(p, q, q')\psi_C(p - q')\tilde{\chi}'_{0D}(q')|\rangle \quad (20)$$

Here we have written

$$\chi_{0B}(p_1) = \theta(p_1^0)\delta(p_1^2 - M^2)\tilde{\chi}'_{0B}(p_1) \tag{21}$$

which is possible since χ_0 obeys the Klein-Gordon equation. We have used the shorthand $dM(q)$ for the invariant measure $d^4q\theta(q_0)\delta(q^2 - M^2)$. The tensors R_{AB}^{CD} and S_{AB}^{CD} are defined as follows:

$$R_{AB}^{CD} = \delta_{AB}^{CD}(\tfrac{1}{2}(p_1+q)\cdot p - M^2) + \tfrac{1}{2}\delta_{AB}^{DC}(p_1-q)^2 + \delta_B^D I(p, p_1-q)_A{}^C + \delta_A^D I(p_1-q, p-q)_B{}^C + \delta_B^C I(p-p_1, q-p_1)_A{}^D \tag{22}$$

and

$$S_{AB}^{CD}(p_1,q,q') = \delta_{AB}^{CD}(M^2 + \tfrac{1}{2}(p_1\cdot(q'-q) - q.q')) + \delta_{AB}^{DC}(\tfrac{1}{2}q\cdot(p_1+q') - M^2) + \delta_A^C I(q-p_1, q-q')_B{}^D - \delta_A^D I(q-p_1, q)_B{}^C - \delta_B^C I(q, q-q')_A{}^D \tag{23}$$

$I(p,q)$ is defined as:

$$I(p,q)_A{}^B = p_{AA'}q^{BA'} - \tfrac{1}{2}\delta_A^B(p\cdot q) \tag{24}$$

This is traceless, linear in p and q and has the property that $I(p,p) = 0$. Hence $I(p,q) = -I(q,p)$. It can be written as

$$I(p,q)_A{}^B = \tfrac{1}{2}(q^0\mathbf{p} - p^0\mathbf{q} - i\mathbf{p}\wedge\mathbf{q})\cdot\tau_A{}^B \tag{25}$$

where τ are the Pauli matrices.

To associate the states with a time, we Fourier transform back just the time component of the fields. Thus

$$|t;\mathbf{p};\mathbf{p}_1;{}_{AB}\rangle = \psi_A(t;\mathbf{p}-\mathbf{p}_1)\tilde{\chi}_{0B}(t;\mathbf{p}_1)|\rangle = \int dp_0 dp_1^0 e^{ip_0t}\psi_A(p-p_1)\tilde{\chi}_{0B}(p_1)|\rangle = \int dp_0 \frac{e^{ip_0t}}{2\sqrt{\mathbf{p}_1^2+M^2}}\psi_A(p-p_1)\tilde{\chi}'_{0B}(p_1)|\rangle\Big|_{p_1 \text{on shell}} \tag{26}$$

For the purpose of calculating the inner product, we approximate ψ with its free-field value, giving

$$\langle t';\mathbf{p}';\mathbf{p}'_1;{}_{A'B'}|t;\mathbf{p};\mathbf{p}_1;{}_{AB}\rangle = \delta(\mathbf{p}-\mathbf{p}')\delta(\mathbf{p}_1-\mathbf{p}'_1)\frac{e^{i\left(\sqrt{\mathbf{p}_1^2+M^2}+\sqrt{(\mathbf{p}-\mathbf{p}_1)^2+m^2}\right)(t-t')}}{2\sqrt{\mathbf{p}_1^2+M^2}\sqrt{(\mathbf{p}-\mathbf{p}_1)^2+m^2}}(p-p_1)_{AA'}p_{1BB'} \tag{27}$$

Thus the states

$$|t;\mathbf{p};\mathbf{p}_1;{}_{ij}\rangle = \frac{2}{\sqrt{mM}}\left(\frac{\mathbf{p}_1^2+M^2}{(\mathbf{p}-\mathbf{p}_1)^2+m^2}\right)^{\frac{1}{4}} L_i{}^A(p-p_1)L_j{}^B(p_1)|t;\mathbf{p};\mathbf{p}_1;{}_{AB}\rangle \tag{28}$$

are orthonormal, i.e.

$$\langle t;\mathbf{p}';\mathbf{p}'_1;{}_{kl}|t;\mathbf{p};\mathbf{p}_1;{}_{ij}\rangle = \delta(\mathbf{p}-\mathbf{p}')\delta(\mathbf{p}_1-\mathbf{p}'_1)\delta_{ik}\delta_{jl} \tag{29}$$

where

$$L_i{}^A(q) = \frac{1}{\sqrt{2m_0(q_0+m_0)}}(q_0+m_0+\mathbf{q}\cdot\tau)_i{}^A \tag{30}$$

with $q^2 = m_0^2$, is the boost that transforms to a frame where q is at rest. Thus

$$((p-p_1)^2-m^2)(\mathbf{p}_1^2+M^2)^{\frac{1}{4}}((\mathbf{p}-\mathbf{p}_1)^2+m^2)^{\frac{1}{4}}L^{-1}{}_A{}^i(p-p_1)L^{-1}{}_B{}^j(p_1)|t;p;p_1;{}_{ij}\rangle = 4e^2\int\frac{dM(q)}{(q-p_1)^2-\mu^2}R_{AB}^{CD}(p,p_1,q)(\mathbf{q}^2+M^2)^{\frac{1}{4}}((\mathbf{p}-\mathbf{q})^2+m^2)^{\frac{1}{4}}L_C^{-1k}(p-q)L^{-1}{}_D{}^l(q)|t;p;q;{}_{kl}\rangle - 8e^4\int\frac{dM(q)}{(q-p_1)^2-\mu^2}\frac{dM(q')}{(q-q')^2-\mu^2}S_{AB}^{CD}(p,q,q')(\mathbf{q}'^2+M^2)^{\frac{1}{4}}((\mathbf{p}-\mathbf{q}')^2-m^2)^{\frac{1}{4}} L^{-1}{}_C{}^k(p-q')L^{-1}{}_D{}^l(q)|t;p;q';{}_{kl}\rangle \tag{31}$$

where $p_1^0 = \sqrt{\mathbf{p}_1^2 + M^2}$ and p_0 is now the operator $-i\partial/\partial t$.

We now consider the case of non-relativistic motion. Put $\mathbf{p} = 0$ and assume that $\mathbf{p}_1 << M$. Also, ignore the e^4 contribution. The Lorentz transformations here are then approximately the identity and R_{AB}^{CD} approximates to $mM\delta_{AB}^{CD}$:

$$\left(\left(p^0 - \sqrt{\mathbf{p}_1^2 + M^2}\right)^2 - \mathbf{p}_1^2 - m^2\right) |t; p; p_1;_{ij}\rangle \approx 4e^2 \int \frac{d^3\mathbf{q}}{2\sqrt{\mathbf{q}^2 + M^2}((q - p_1)^2 - \mu^2)} mM |t; p; q;_{ij}\rangle \tag{32}$$

Now

$$(q - p_1)^2 = \left(\sqrt{\mathbf{q}^2 + M^2} - \sqrt{\mathbf{p}_1^2 + M^2}\right)^2 - (\mathbf{q} - \mathbf{p}_1)^2 \approx \left(\frac{\mathbf{q}^2}{2M} - \frac{\mathbf{p}_1^2}{2M}\right)^2 - (\mathbf{q} - \mathbf{p}_1)^2 \approx -(\mathbf{q} - \mathbf{p}_1)^2 \tag{33}$$

So

$$\left(p^0 - \sqrt{\mathbf{p}_1^2 + M^2} - \sqrt{\mathbf{p}_1^2 + m^2}\right)\left(p^0 - \sqrt{\mathbf{p}_1^2 + M^2} + \sqrt{\mathbf{p}_1^2 + m^2}\right) |t; p; p_1;_{ij}\rangle \approx$$
$$-2e^2 m \int \frac{d^3\mathbf{q}}{(\mathbf{q} - \mathbf{p}_1)^2 + \mu^2} |t; p; q;_{ij}\rangle \tag{34}$$

So, at our level of approximation

$$\left(p^0 - m - M - \frac{\mathbf{p}_1^2}{2m_r}\right) |t; p; p_1;_{ij}\rangle = -e^2 \int \frac{d^3\mathbf{q}}{(\mathbf{q} - \mathbf{p}_1)^2 + \mu^2} |t; p; q;_{ij}\rangle \tag{35}$$

where $m_r = mM/(m + M)$ is the reduced mass.

We may use the fact that $-i\hat{\mathbf{x}}$ is the momentum displacement operator on account of the fact that $[x_i, p_j] = i\delta_{ij}$. Thus

$$|\mathbf{q}\rangle = e^{-i(\mathbf{q} - \mathbf{p}_1)\cdot\hat{\mathbf{x}}} |\mathbf{p}_1\rangle \tag{36}$$

Hence

$$\left(p^0 - m - M - \frac{\mathbf{p}_1^2}{2m_r} + e^2 \int \frac{d^3\mathbf{q}}{(\mathbf{q} - \mathbf{p}_1)^2 + \mu^2} e^{-i(\mathbf{q} - \mathbf{p}_1)\cdot\hat{\mathbf{x}}}\right) |t; p; p_1;_{ij}\rangle = 0 \tag{37}$$

Now

$$\int d^3\mathbf{q} \frac{e^{-i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 + \mu^2} = \frac{2\pi^2}{|\mathbf{x}|} e^{-\mu|\mathbf{x}|} \tag{38}$$

Hence

$$\left(p^0 - m - M - \frac{\mathbf{p}_1^2}{2m_r} + e^2 \frac{2\pi^2}{|\hat{\mathbf{x}}|} e^{-\mu|\hat{\mathbf{x}}|}\right) |t; p; p_1;_{ij}\rangle = 0 \tag{39}$$

With the conventions here, the fine-structure constant is defined by $\alpha = 2\pi^2 e^2$, giving us

$$\left(p^0 - m - M - \frac{\mathbf{p}_1^2}{2m_r} + \frac{\alpha}{|\hat{\mathbf{x}}|} e^{-\mu|\hat{\mathbf{x}}|}\right) |t; p; p_1;_{ij}\rangle = 0 \tag{40}$$

If we put the photon mass μ to zero, this then leads to the usual gross structure of a single-electron atom.