STÜCKELBERG COVARIANT PERTURBATION THEORY IN 5 MINUTES

C.G. Oakley, 17 February 2012 (last modified 30 September 2019)

The equation of motion of Φ^3 scalar field theory is

$$(\partial^2 + m^2)\Phi(x) = -\lambda\Phi^2(x) \tag{1}$$

The standard technique is to create an interaction picture time-evolution operator that acts on quasi-free field states identified as incoming particles in a scattering experiment, and to work out the matrix element with quasi-free field states identified as outgoing particles. Implicitly, this is a Maclaurin series in the coupling constant, but Stückelberg's method just does this directly:

$$\Phi(x) = \Phi_0(x) + \lambda\Phi_1(x) + \lambda^2\Phi_2(x) + \cdots \tag{2}$$

When this is substituted back into (1) one gets an infinite number of equations as follows:

$$\begin{aligned}
(\partial^2 + m^2)\Phi_0(x) &= 0 \\
(\partial^2 + m^2)\Phi_1(x) &= -\Phi_0(x)^2 \\
(\partial^2 + m^2)\Phi_2(x) &= -(\Phi_0(x)\Phi_1(x) + \Phi_1(x)\Phi_0(x)) \\
&\vdots \\
(\partial^2 + m^2)\Phi_r(x) &= -\sum_{i=0}^{r-1}\Phi_i(x)\Phi_{r-i-1}(x)
\end{aligned} \tag{3}$$

Evidently $\Phi_0(x)$ is a free field, and the higher order Φ's are directly or indirectly determined from this, although each is arbitrary to the extent of addition of a free-field solution. Note that Φ_0 is a formal construction here—no attempt is being made to identify it with "in" or "out" scattering states.

As a side note, an expansion of this kind, regardless of the particular form of the self-interaction, can be substituted into the equal-time commutators

$$\begin{aligned}
[\Phi(\mathbf{x}, t), \dot{\Phi}(\mathbf{x}', t)] &= i\delta(\mathbf{x} - \mathbf{x}') \\
[\Phi(\mathbf{x}, t), \Phi(\mathbf{x}', t)] &= 0
\end{aligned} \tag{4}$$

from which we find that Φ_0 has the standard equal-time commutator. Higher orders can be solved—Haag attempted something like this in his 1955 paper, inserting an expansion in terms of free fields with no assumptions about the interacting equation of motion, and the indications were that local field equations, such as Φ^3 theory here were what was needed to solve to all orders.

The negative-energy part of Φ_0 must annihilate the vacuum, even in the presence of interactions, since we would otherwise have negative-energy states.

These simple things, namely the reduction of the interacting field in terms of Φ_0, the knowledge that Φ_0 has the expected commutator function and the knowledge that its negative-energy part annihilates the vacuum, give one a framework for calculating matrix elements.

One finds, from second-order upwards, pathological infinities arising when one commutes the negative-energy parts past to annihilate the vacuum.

If, on the other hand, one's starting point was

$$(\partial^2 + m^2)\Phi(x) = -\lambda :\Phi^2(x): \tag{5}$$

instead of (1) (i.e. one was allowed to place the terms from lower order in normal order), then these would not appear: there are still loop integrals, but these do not diverge.

Whether the normal-ordered version still solves the commutators (4) is not clear, but it seems likely.

Being able to calculate matrix elements is not, however, the end of the story because of the disquieting property that

$$\langle 0|\Phi(t,\mathbf{x})\Phi(t,\mathbf{x}')\Phi(t,\mathbf{x}'')|0\rangle \neq 0 \tag{6}$$

In other words, a two-particle state at time t is not orthogonal to a one-particle state at the same time! This, in fact, is a direct consequence of Haag's theorem, which, crudely stated, says that there is no interaction picture (if there was, we could unitarily transform the fields in (6) into free fields, whereupon the matrix element would vanish identically).

It would appear, though, that despite this, the undesired behaviour takes the form of transients. The loop integrals seem also to lead just to transients—the contributions that survive for large time elapse between initial and final states can be identified by poles in the amplitude, and these give the same expressions as tree-level Feynman graphs.

The situation at present may be summarised as follows:

- Unlike Feynman-Dyson perturbation theory, this is not merely a scattering theory, as the time variable is not eliminated.
- As a scattering theory, up to tree level, for Φ^3 scalar field theory, and for quantum electrodynamics, the approach gives the same answers as the corresponding Feynman graphs.
- At higher order, like Feynman-Dyson perturbation theory, there are loops, and these are generally divergent integrals.
- A formulation is possible without any divergent integrals, but the equations of motion are no longer the simple ones of Φ^3 theory and quantum electrodynamics.

Stückelberg Covariant Perturbation Theory simplifies things a little but cannot in itself be the answer because of the power series in the coupling.

Using a slightly different approach, originally pioneered by Bethe and Salpeter, one may obtain the Dirac equation for the non-relativistic two-body system from the equations of quantum electrodynamics. My version of the argument is given here: http://cgoakley.org/qft/2bodywe3.pdf.

The thing I would like to draw the reader's attention to is that although approximations are made, there are no infinite subtractions.

On the other hand if one tried to get to the same place with the further assumption that it was possible to write the interacting fields as Maclaurin series in the coupling, one would, by second order, be contending with fermion and photon self-energies, both of which are divergent integrals—see here: http://cgoakley.org/qft/qedwip.pdf, or, for that matter, any quantum field theory text book.

So, some kind of generalisation of the Bethe-Salpeter equation to cover all processes looks like a good way to go.

Tampering with the infinite terms in the power series, however, is quite obviously *not* a good way to go, and from the point at which they start to do this, text books on quantum field theory lose the right to claim that this is, in truth, their subject matter.