THE GROUND STATE OF A TWO-BODY BOUND STATE IN RELATIVISTIC FORM

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In non-relativistic quantum mechanics, the energy-levels of a bound state comprising of a particle of mass m_1 and charge $+e$ with a particle of mass m_2 and charge $-e$ are obtained by solving an energy eigenvalue equation with Hamiltonian

$$H = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{\alpha}{|\mathbf{x}_1 - \mathbf{x}_2|} \tag{1.1}$$

where $\alpha = e^2$ is the fine structure constant. To do this we change to center-of-mass and relative co-ordinates:

$$\mathbf{X} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{M}; \qquad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \tag{1.2}$$

$$\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2; \qquad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{M} \tag{1.3}$$

Where $M = m_1 + m_2$. It is easy to check that

$$[X_i, P_j] = i\delta_{ij}; \quad [x_i, P_j] = 0; \quad [x_i, p_j] = i\delta_{ij} \quad \text{and} \quad [X_i, p_j] = 0 \tag{1.4}$$

as required. The Hamiltonian is then

$$H = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{|\mathbf{x}|} \tag{1.5}$$

where $m_r = m_1 m_2/M$ is the reduced mass. As is well known the solution for the overall wavefunction is plane waves with the dispersion relation $E = P^2/(2M)$, and for the relative wavefunction, a set of discrete energy levels given by

$$E_n = -\frac{m_r\alpha^2}{2n^2} \tag{1.6}$$

where $n = 1, 2, 3, \cdots$ is called the principal quantum number. The ground state ($n = 1$) wavefunction is

$$\psi(\mathbf{x}) = \pi^{-1/2}(m_r\alpha)^{3/2} e^{-m_r\alpha|\mathbf{x}|} \tag{1.7}$$

The wavefunction in momentum space is defined to be

$$\psi(\mathbf{p}) = (2\pi)^{-3/2} \int d^3\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}} \psi(\mathbf{x}) \tag{1.8}$$

Which for this wavefunction can be calculated as

$$\psi(\mathbf{p}) = \frac{\sqrt{8m_r^5\alpha^5}}{\pi} (\mathbf{p}^2 + m_r^2\alpha^2)^{-2} \tag{1.9}$$

To find the relativistic form of this, consider any field operator—free *or* interacting—$\Phi(x)$. The definition of P_a as the translation generators requires

$$[P_a, \Phi(x)] = -i\partial_a \Phi(x) \tag{1.10}$$

from which it follows that the four-dimensional Fourier transform defined by

$$\Phi(p) = (2\pi)^{-4} \int d^4x\, e^{-ip\cdot x} \Phi(x) \tag{1.11}$$

will obey

$$[P_a, \Phi(p)] = p_a \Phi(p) \tag{1.12}$$

Thus $|p\rangle = \Phi(p)|0\rangle$ is a state of four-momentum p; to get a state of three-momentum $\mathbf{p}$ associated with time t we need to Fourier transform back just the time component, i.e.

$$|t;\mathbf{p}\rangle = \int dp_0\, e^{ip_0 t}|p\rangle \tag{1.13}$$

Ignoring the normalisation constant, the $n = 1$ bound state, at rest, is thus given by

$$|t;\mathbf{0}\ [1s]\rangle = \int e^{i(p_1^0+p_2^0)t} dp_1^0 dp_2^0 \frac{d^3\mathbf{p_1}}{(\mathbf{p_1}^2 + m_r^2\alpha^2)^2}|p_1, (p_2^0, -\mathbf{p}_1)\rangle \tag{1.14}$$

Note that we are also ignoring the fact that the normalisation of the states (1.13)—assumed to be dominated by the free-field contribution—will have an multiplicative factor of $2E(\mathbf{p})$ relative to that for (1.8), since in the first approximation this factor is a constant $(2m)$.

Fourier-transforming with respect to time for the combined system then gives one

$$|p;\ [1s]\rangle = \int \frac{d^4p_1}{(\mathbf{p_1}^2 + m_r^2\alpha^2)^2}|p_1, p - p_1\rangle \tag{1.15}$$

where $p = (p^0, \mathbf{0})$. In the first approximation, the component particles are on-shell, i.e. $p_1^2 = m_1^2$ and $(p - p_1)^2 = m_2^2$, so

$$\begin{aligned} p_0 &= \sqrt{\mathbf{p}_1^2 + m_1^2} + \sqrt{\mathbf{p}_1^2 + m_2^2} \\ &\approx m_1 + m_2 + \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_1^2}{2m_2} = m_1 + m_2 + \frac{\mathbf{p}_1^2}{2m_r} \end{aligned} \tag{1.16}$$

So

$$p^2 = p_0^2 = (m_1 + m_2)^2 + 2(m_1 + m_2)\frac{\mathbf{p}_1^2}{2m_r} + O(\mathbf{p_1}^4) \tag{1.17}$$

Thus

$$\mathbf{p}_1^2 \approx \frac{m_r}{m_1 + m_2}(p^2 - (m_1 + m_2)^2) \tag{1.18}$$

and hence

$$|p;\ [1s]\rangle = \int \frac{d^4p_1}{(\frac{m_r}{m_1+m_2}(p^2 - (m_1 + m_2)^2) + m_r^2\alpha^2)^2}|p_1, p - p_1\rangle \tag{1.19}$$

Ignoring the overall scaling this gives us

$$|p;\ [1s]\rangle = \int \frac{d^4p_1}{(p^2 - m_{1\mathrm{s}}^2)^2}|p_1, p - p_1\rangle \tag{1.20}$$

where

$$m_{1\mathrm{s}}^2 = (m_1 + m_2)^2 - m_1 m_2\alpha^2 \tag{1.21}$$

which is now properly relativistic. Note (i) that $(p^2 - m_{1\mathrm{s}}^2)^{-2}$ can be taken outside the integral, i.e.

$$|p;\ [1s]\rangle = \frac{1}{(p^2 - m_{1\mathrm{s}}^2)^2} \int d^4p_1 |p_1, p - p_1\rangle \tag{1.22}$$

and (ii)

$$\begin{aligned} m_{1\mathrm{s}} =&\ (m_1 + m_2)\sqrt{1 - \frac{m_1 m_2}{(m_1 + m_2)^2}\alpha^2} \\ \approx&\ m_1 + m_2 - \frac{1}{2}(m_1 + m_2)\frac{m_1 m_2}{(m_1 + m_2)^2}\alpha^2 \\ \approx&\ m_1 + m_2 - \frac{1}{2}m_r\alpha^2 \end{aligned} \tag{1.23}$$

so the pole in the expression, although unreachable for on-shell particles, is at the mass-energy of the bound state.

Note that this is not the only relativistic expression that reproduces the non-relativistic behaviour correctly; the factor $p^2/(m_1+m_2)^2$ is in this level of approximation equal to unity, and such factors may be judiciously inserted without spoiling the non-relativistic limit.

One may check that a pole at the invariant mass of the bound state is also a feature for excited states.

This result could be of use in applying Stueckelberg covariant perturbation theory to bound states.

Take the decay of positronium in the rest frame: $e^+e^- \to \gamma\gamma(\gamma)$. The Feynman graph calculation shows that the invariant mass of the photons cannot be less than $2m_e$. Yet we know that it is—not by much, admittedly, but there is still the binding energy to be taken account of, and so the Feynman amplitude is consequently incorrect.

It will be noted that the energy conservation delta function in Stueckelberg covariant perturbation theory is in fact an approximation of a pole and a phase—the same approximation one makes in deriving Fermi's Golden Rule. Although the naive perturbation theory rules out composite states with invariant mass below that of the components, a useful line of enquiry would be to see if a pole in the wavefunction itself, such as we have here, allows one to get round this, thereby giving us a relativistic description of bound states.