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ON THE POSSIBILITY OF QUANTUM FIELD THEORY WITHOUT RENORMALISATION

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The requirements of positive energy and a Poincaré-invariant vacuum, strictly applied, seem to lead to a quantum field theory that does not need renormalisation. This is illustrated for the case of ϕ^4 scalar field theory.

Feynman-Dyson perturbation theory, although undoubtedly an elegant and powerful structure has (in my opinion) two major flaws: (i) it is difficult to see how the definition of the vacuum state, namely that it is annihilated by the interaction-picture annihilation operators, is Poincaré invariant, since the definition of the interaction picture requires a preferential choice of timelike direction; and (ii) infinite answers are obtained for most physical observables: only by introducing a set of rules which involve infinite subtractions can these be made finite.

The thesis of this letter is that these two defects are closely related, and to be more specific, the ruthless application of the principles of positive energy and Poincaré invariance of the vacuum leads to a theory which does not need renormalisation. Our starting point is the equation of motion of ϕ^4 theory:

$$(\partial^2 + m^2)\phi(x) + \frac{\lambda}{6}\phi^3(x) = 0 \tag{1}$$

We need to inject here the additional assumption that the solutions of physical interest, as functions of λ, are continuous and infinitely differentiable at $\lambda = 0$. This allows us to make the expansion

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots \tag{2}$$

(1) is then an infinite set of equations, one for each power of λ. If we define $\phi(p)$ by

$$\phi(p) = (2\pi)^{-4}\int d^4x\, e^{-ip\cdot x}\, \phi(x) \tag{3}$$

then we find that the equations of motion are

$$(p^2 - m^2)\phi_0(p) = 0 \tag{4}$$

$$(p^2 - m^2)\phi_1(p) = \frac{1}{6}\int d^4p_1 d^4p_2 d^4p_3\ \delta(p - p_1 - p_2 - p_3)\ \phi_0(p_1)\phi_0(p_2)\phi_0(p_3) \tag{5}$$

$$\begin{aligned}(p^2 - m^2)\phi_2(p) = \frac{1}{6}\int d^4p_1 d^4p_2 d^4p_3\ \delta(p - p_1 - p_2 - p_3)\ \big[\phi_1(p_1)\phi_0(p_2)\phi_0(p_3) \quad + \\ \phi_0(p_1)\phi_1(p_2)\phi_0(p_3) + \phi_0(p_1)\phi_0(p_2)\phi_1(p_3)\big]\end{aligned} \tag{6}$$

etc.

Evidently ϕ_0 satisfies free-field equations of motion. The higher-order terms are determined by ϕ_0, and we can write (e.g.)

$$\phi_1(p) = c_1.\phi_0(p) + \frac{1}{p^2 - m^2}\cdot\frac{1}{6}\cdot\int d^4p_1 d^4p_2 d^4p_3\ \delta(p - p_1 - p_2 - p_3)\ \phi_0(p_1)\phi_0(p_2)\phi_0(p_3) \tag{7}$$

(it is not necessarily obvious that $p^2 - m^2$ can be inverted, but we will assume it here).

Since the theory is determined by $\phi_0(p)$ it makes sense to evaluate the commutators of these quantities. We note that

$$f(\lambda).\phi_0(p) = \phi(p) - \frac{1}{p^2 - m^2} \cdot \frac{\lambda}{6} \cdot \int d^4p_1 d^4p_2 d^4p_3 \; \delta(p - p_1 - p_2 - p_3) \; \phi(p_1)\phi(p_2)\phi(p_3) \tag{8}$$

where

$$f(\lambda) = 1 + c_1\lambda + c_2\lambda^2 + \cdots \tag{9}$$

However the definitions of "in" and "out" fields are given by

$$\phi_{\substack{\text{in}\\ \text{out}}}(x) = \phi(x) + \frac{\lambda}{6} \int d^4x' \Delta_{\substack{\text{ret}\\ \text{adv}}}(x - x')\phi^3(x') \tag{10}$$

which in momentum space are

$$\phi_{\substack{\text{in}\\ \text{out}}}(p) = \phi(p) - \frac{\lambda}{6} \cdot \frac{1}{p^2 - m^2 \mp i\zeta.\epsilon(p^0)} \cdot \int d^4p_1 d^4p_2 d^4p_3 \; \delta(p - p_1 - p_2 - p_3) \; \phi(p_1)\phi(p_2)\phi(p_3) \tag{11}$$

where ζ is an infinitesimal positive number and $\epsilon(t) = \text{sign}(t)$. We shall find here that the parameter ζ is unimportant, so effectively

$$f(\lambda).\phi_0(p) = \phi_{\text{in}}(p) = \phi_{\text{out}}(p) \tag{12}$$

The commutators of ϕ_{in} and ϕ_{out} have been determined by Zimmermann: the canonical commutators of the interacting field imply that ϕ_{in} and ϕ_{out} have free-field commutators. Thus $f(\lambda) = 1$ and

$$[\phi_0(p), \phi_0(q)] = (2\pi)^{-3}\delta(p + q)\delta(q^2 - m^2)\epsilon(q^0) \tag{13}$$

Finally we need some information about the vacuum state: Poincaré invariance implies that

$$[P_\mu, \phi_0(x)] = -i\partial_\mu\phi_0(x) \tag{14}$$

Hence

$$[P_\mu, \phi_0(p)] = p_\mu\phi_0(p) \tag{15}$$

Since the vacuum is Poincaré invariant we can write

$$P^\mu\phi_0(p)|0\rangle = [P^\mu, \phi_0(p)]|0\rangle = p^\mu\phi_0(p)|0\rangle \tag{16}$$

so $\phi_0(p)|0\rangle$ is a state of four-momentum p_μ. However we also require that there are no negative-energy states, so $\phi_0(p)|0\rangle$ must be simply zero if $p_0 < 0$. This is all we need to do calculations.

Consider first the matrix element

$$\langle 0|\phi(p)\phi(q)|0\rangle \tag{17}$$

We can expand the fields in powers of λ and so obtain

$$\begin{aligned}\langle 0|&\left(\phi_0(p) + \frac{\lambda}{6}\frac{1}{p^2 - m^2}\int d^4p_1 d^4p_2 \; \phi_0(p_1)\phi_0(p_2)\phi_0(p - p_1 - p_2) + \cdots\right)\\ &\qquad\left(\phi_0(q) + \frac{\lambda}{6}\frac{1}{q^2 - m^2}\int d^4q_1 d^4q_2 \; \phi_0(q_1)\phi_0(q_2)\phi_0(q - q_1 - q_2) + \cdots\right)|0\rangle\end{aligned} \tag{18}$$

The lowest-order contribution is

$$(2\pi)^{-3}\delta(p + q)\delta(q^2 - m^2)\theta(q^0) \tag{19}$$

The higher-order corrections have infinities which can be removed by normal ordering: consider

$$\int d^4q_1 d^4q_2 \phi_0(q_1)\phi_0(q_2)\ \phi_0(q-q_1-q_2)|0\rangle \tag{20}$$

A little algebra shows that this is of the form

$$\int d^4q_1 d^4q_2\ \phi_0^{(+)}(q_1)\phi_0^{(+)}(q_2)\phi_0^{(+)}(q-q_1-q_2)|0\rangle + (2\pi)^{-3}.3\Big[\int d^4p_1\ \delta(p_1^2-m^2)\theta(p_1^0)\Big]\phi_0^{(+)}(p)|0\rangle \tag{21}$$

where

$$\phi_0^{(+)}(q) = \theta(q^0)\phi_0(q) \tag{22}$$

is the positive-energy part of ϕ_0, which creates a particle from the vacuum (N.B. (i) the positive-energy parts of the field operators commute with each other; (ii) positive energy is equivalent to negative frequency and vice versa).

Thus we have resolved this state into a one-particle state and a three-particle state. Since $p^2 > 9m^2$ for a three-particle state, there is no difficulty with applying $(p^2-m^2)^{-1}$ to the left to obtain $\phi_1(p)$: however for the one-particle part, not only is (p^2-m^2) not invertible, but also there is a divergent constant multiplying the expression. These difficulties can be removed by normal ordering the interaction in (1): with the annihilation part to the right $\phi_1(p)$ can then create the three-particle part only (but I am not very happy with this since the interaction is then non-local: but it seems that for theories of evident physical application, such as quantum electrodynamics, under certain conditions normal ordering is not necessary– work is being done on this). The normal ordering ensures that the states $\phi_n(p)|0\rangle$ contain no single-particle parts for $n > 0$. Hence the lowest-order correction to (17) is

$$\begin{aligned}\lambda^2\langle 0|\phi_1(p)\phi_1(q)|0\rangle = \big(\frac{\lambda}{6}\big)^2 \frac{1}{p^2-m^2}\frac{1}{q^2-m^2}\int d^4p_1 d^4p_2 d^4q_1 d^4q_2 \\ \langle 0|\phi_0^{(-)}(p_1)\phi_0^{(-)}(p_2)\phi_0^{(-)}(p-p_1-p_2)\phi_0^{(+)}(q_1)\phi_0^{(+)}(q_2)\phi_0^{(+)}(q-q_1-q_2)|0\rangle\end{aligned} \tag{23}$$

$$\begin{aligned}= \frac{\lambda^2}{6}\frac{1}{p^2-m^2}\frac{1}{q^2-m^2}(2\pi)^{-9}\int d^4q_1 d^4q_2\ \theta(q_1^0)\delta(q_1^2-m^2) \\ \theta(q_2^0)\delta(q_2^2-m^2)\theta(q^0-q_1^0-q_2^0)\delta((q-q_1-q_2)^2-m^2)\delta(p+q)\end{aligned} \tag{24}$$

There is a diagram interpretation following directly from this calculational technique: to calculate the matrix element

$$\begin{aligned}&\langle 0|\phi(q_1)\phi(q_2)\cdots\phi(p_1)\phi(p_2)\cdots|0\rangle \\ &q_1^0, q_2^0, \cdots < 0 \quad \text{(outgoing particles)} \\ &p_1^0, p_2^0, \cdots > 0 \quad \text{(incoming particles)}\end{aligned} \tag{25}$$

we sum an infinite series of diagrams as follows:

(1) All topologically possible diagrams are legitimate provided a line comes in for incoming particles, possibly connecting to some internal network, and then goes out so that all external lines are accessed. No disconnected diagrams are possible.

(2) The network is drawn such that four-momentum is conserved at each vertex. There is an overall momentum-conservation factor $\delta(q_1+q_2+\cdots+p_1+p_2+\cdots)$. A factor λ is placed at each vertex (and these are always four-point), and for each loop a factor $\int d^4q$ for the undetermined momentum.

(3) There are two kinds of propagator:

(i) Type 1 ("proliferator"). Drawn with a heavy line. Factor $(p^2-m^2)^{-1}$. These are always connected through other proliferators to an external line. However (a) it is never possible to go from one external line to another by proliferators alone, and (b) one may not have a loop of just proliferators.

(ii) Type 2 ("propagator"). Drawn with a thin line. Factor $(2\pi)^{-3}\theta(p^0)\delta(p^2 - m^2)$ May connect to external lines. A vertex may not consist solely of propagators.

(4) Divide loops with the occurence of g identical propagators by $g!$.

(5) Draw arrows starting from each incoming line, outwards from a vertex to show momentum flow. Possible vertices are shown in fig. 1.

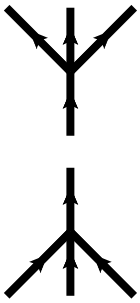

Fig. 1: Types of interaction vertex.

This is best illustrated by examples. (23) is the second-order self energy (fig. 2):

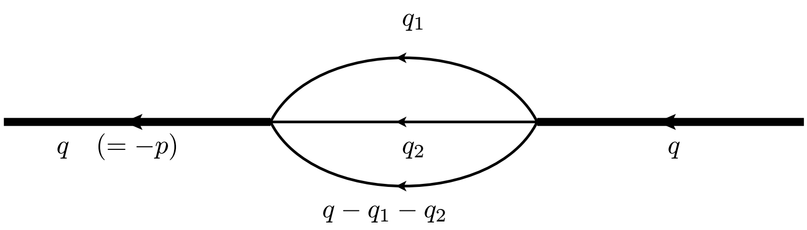

Fig. 2: Second-order self energy.

Observing the graph rules we have:

incoming and outgoing proliferators: $\frac{1}{p^2-m^2}\frac{1}{q^2-m^2}$

three propagators: $(2\pi)^{-3}\theta(q_1^0)\delta(q_1^2-m^2)(2\pi)^{-3}\theta(q_2^0)\delta(q_2^2-m^2)(2\pi)^{-3}\theta(q^0-q_1^0-q_2^0)\delta((q-q_1-q_2)^2-m^2)$

undetermined momenta: $\int d^4q_1 d^4q_2$

two vertices: λ^2

statistical factor: $\frac{1}{3!}$

momentum conservation: $\delta(q+p)$

which gives us (24):

$$\frac{\lambda^2}{6}\frac{1}{p^2-m^2}\frac{1}{q^2-m^2}\delta(p+q)(2\pi)^{-9}\int d^4q_1 d^4q_2\ \theta(q_1^0)\delta(q_1^2-m^2)$$
$$\theta(q_2^0)\delta(q_2^2-m^2)\theta(q^0-q_1^0-q_2^0)\delta((q-q_1-q_2)^2-m^2)$$

Similarly we could draw graphs for two-body scattering, e.g. Fig. 3:

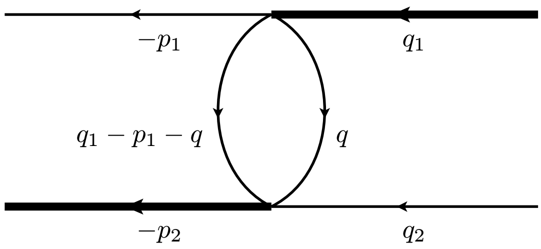

Fig. 3: Two-body scattering graph.

which gives us

$$\delta(p_1+p_2+q_1+q_2)\frac{1}{q_1^2-m^2}(2\pi)^{-6}\theta(-p_1^0)\delta(p_1^2-m^2)\theta(q_2^0)\delta(q_2^2-m^2)\frac{1}{p_2^2-m^2}$$
$$\frac{\lambda^2}{2}\int d^4q(2\pi)^{-6}\theta(q^0)\delta(q^2-m^2)\theta(q_1^0-p_1^0-q^0)\delta((q_1-p_1-q)^2-m^2) \quad (26)$$

Note that the graph rules are such that the graphs give numerical expressions which are not infinite.

The physical interpretation is fairly evident. The state

$$|t,\mathbf{p}\rangle = \int_0^\infty dp^0\ e^{ip_0t}\ \phi(p)|0\rangle \quad (27)$$

represents a particle at time t, with three-momentum $\mathbf{p}$. The states of time t and different $\mathbf{p}$ are orthogonal, and can be made orthonormal, by multiplying by a suitable function of $\mathbf{p}$. This becomes a little more complicated for multiparticle states because we find that higher-order corrections imply that although states of different total three momentum are orthogonal, states of the same total momentum but different individual momenta are not. This is not surprising since the argument that leads us to suppose that they would be orthogonal is that they are eigenvectors of Hermitian operators with different eigenvalues: but the operators corresponding to individual particle momenta simply do not exist and therefore this does not apply.

The same methods can be applied to quantum electrodynamics, or indeed any field theory (I have chosen ϕ^4 theory simply because it avoids complications arising from gauge freedom).

Reference

W. Zimmermann, Nuovo Cimento **10** (1958) 597.