

QUANTUM ELECTRODYNAMICS WITHOUT THE INTERACTION PICTURE

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This paper presents a new, axiomatic formulation of quantum electrodynamics which is consistent with Haag's theorem, i.e. it does not require the interaction picture. The method is based on a power series expansion of the interacting fields in the coupling constant, from which the amplitudes for physical processes can be obtained, from first principles, by inspection of the terms and comparison with time-dependent perturbation theory. Up to "tree" level we reproduce the results obtained from Feynman graph analysis.

For higher orders, we find divergent integrals, as is the case with other formulations of quantum field theory. Within this method, these infinities can be treated without recourse to the renormalisation procedure, which is an improvement on the traditional methods, but cannot be the final answer.

Apart from this, the benefits of this method are (i) the time variable is not eliminated from amplitudes, so it is in principle possible to treat processes other than just scattering and (ii) the notion that the vacuum is invariant and the state of lowest energy in any frame of reference generates no contradiction (there are no vacuum bubbles, for example).

1. Introduction

Even now, over sixty years after Quantum Field Theory was first devised, we are confronted with a fundamental problem: interacting relativistic quantum field theory does not exist. The study has long continued on the supposition that the bad mathematics that it is all based upon—principally the renormalisation procedure in all its guises—would be somehow made rigorous by the construction of a completely finite "underlying" theory.

The truth is, however, that no such theory has appeared, and we are scarcely any wiser than Dirac, Heisenberg and Pauli were when, in the late twenties, they first found difficulties with higher order terms involving an electron interacting with a quantised electromagnetic field. The contributions of field theorists since then have been merely to restate this problem rather than to solve it.

The investigations undertaken in this paper do not constitute a final solution, but they do give the author, at least, hope that the problem is not insoluble. The reason for this optimism is that the methods are more precise, and bring the problems into much sharper focus. To be more specific:

(i) The formulation is consistent with Haag's theorem; i.e. one is not required to use the interaction picture—which is fortunate as it can be *proved* that within a relativistic quantum field theory this picture does not exist.

(ii) The formulation has a vacuum state which is *definitely* invariant and *definitely* the state of lowest energy in any frame of reference. There are *no* vacuum graphs.

(iii) One unsatisfactory consequence of the methods (the appearance of infinities in higher order graphs) can be dealt with without any "renormalisation".

A reason for supposing that a completely finite theory lies in this direction is that if we expand out the Green function for the electromagnetic interaction of a proton and electron within ordinary quantum mechanics, with respect to the coupling constant, then we get terms which are infinite, reflecting the fact that this quantity cannot not be expressed as a power series in the coupling constant. It could be, then, that that if we can ever construct the theory presented here without ever making the assumption of power-series expandability in the coupling, all our infinities would disappear.

2. The construction of the theory

2.1. THE AXIOMS; FREE FIELDS

It was shown in a previous paper [2] that a self-interacting scalar field theory could be constructed without the use of the interaction picture, and without requiring renormalisation. The construction of a theory of quantum electrodynamics follows along analogous lines. We base the theory upon the following set of axioms:

- I *The states of a physical system form a linear vector space \mathcal{V} over the complex numbers \mathcal{C} , and this is equipped with a sesquilinear, positive-definite inner product.*
- II *There exists a representation of the orthochronous Poincaré group on \mathcal{V} , which preserves the inner product.*
- III *There exists a normalisable, Poincaré-invariant state $|0\rangle$ called the vacuum.*
- IV *All the eigenvalues of the translation generators, or the four-momentum operator P_a lie on or within the forward light cone.*
- V *There exist linear maps ψ_A and $\phi_{AB} : \mathcal{M} \otimes \mathcal{V} \rightarrow \mathcal{V}$ called respectively the electron field operator and the photon field operator, where \mathcal{M} is Minkowski space. The latter is subject to the constraints*

$$\phi_{[AB]} = 0 \quad \text{and} \quad \partial^{AA'} \phi_{AB} = -\partial_{BB'} \bar{\phi}^{A'B'}. \quad (2.1)$$

The second constraint is known as the “homogeneous Maxwell equations”. The indices are of $SL(2, \mathcal{C})$, the covering group of the Lorentz group.

- VI *A pair of field operators will either commute or anticommute when the spacetime points that they refer to are separated by spacelike intervals. The (anti)commutators of fields referring to the same spacetime point are always c -numbers.*

We have included parity in the symmetry group of the theory, since quantum electrodynamics is known to conserve this. There must be a parity conjugate to the field ψ_A , and this will be of the kind $\bar{\chi}^{A'}$, so it is convenient to group the two into a Dirac spinor thus: $\psi_\alpha = (\psi_A, \bar{\chi}^{A'})$. In the case of the photon field, we form the Lorentz tensor

$$F_{ab} = \epsilon_{A'B'} \phi_{AB} + \epsilon_{AB} \bar{\phi}_{A'B'} \quad (2.2)$$

(using “abstract index” notation) which as such, has the parity operation already defined upon it. The appropriate free field theory emerges when we specify that the particle states associated with each field are irreducible representations of the Poincaré group of spins $\frac{1}{2}$ and 1 respectively. This implies the Klein-Gordon equation

$$(\partial^2 + m^2) \psi_A = 0 \quad (2.3)$$

on the spinor field—which gives the particle states definite mass m . The tensor structure then guarantees that it represents spin $\frac{1}{2}$. We do not apply the complete irreducibility constraint

$$\psi_A = \frac{i\sqrt{2}}{m} \partial_{AA'} \bar{\psi}^{A'} \quad (2.4)$$

since we require two sets of particle states—“electrons” and “positrons”. The parity conjugate spinor $\bar{\chi}^{A'}$ may conveniently be defined from ψ_A through

$$\bar{\chi}^{A'} = \frac{i\sqrt{2}}{m} \partial^{AA'} \psi_A \quad (2.5)$$

(2.3) and (2.5) are then embodied in the Dirac equation

$$(i\partial - m) \psi = 0. \quad (2.6)$$

The axioms of the theory then determine the fact that anticommutators (rather than commutators) reduce to c -numbers and that

$$\begin{aligned} \{\psi_\alpha(x), \bar{\psi}^\beta(x')\} &= i(2\pi)^3 (i\partial + m)_\alpha{}^\beta \Delta(x - x') \\ \{\psi_\alpha(x), \psi_\beta(x')\} &= 0 \end{aligned} \quad (2.7)$$

$$\text{where} \quad \Delta(x) = -i \int \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2) \epsilon(p_0) e^{-ip \cdot x} \quad (2.8)$$

is the usual commutator function.

The normalisations used here, although not conventional, will make the analysis simpler later on. It is more convenient to express the anticommutators in terms of the Fourier transform fields defined by

$$\psi(p) = (2\pi)^{-4} \int d^4 x e^{-ip \cdot x} \psi(x) \quad (2.9)$$

This leads to

$$\begin{aligned} \{\psi_\alpha(p), \bar{\psi}^\beta(q)\} &= (\not{p} - m)_\alpha^\beta \delta(p - q) \delta(p^2 - m^2) \epsilon(p_0) \\ \{\psi_\alpha(p), \psi_\beta(q)\} &= 0. \end{aligned} \quad (2.10)$$

In the case of the photon field, the irreducibility constraint gives

$$\partial^\alpha F_{ab} + \mu^2 A_b = 0 \quad (2.11)$$

where A_b is defined through

$$F_{ab} = \partial_a A_b - \partial_b A_a \quad (2.12)$$

which is the solution of (2.1), (which takes the form

$$\partial_{[a} F_{bc]} = 0 \quad (2.13)$$

in this notation). We find that it is the commutator (rather than the anticommutator) which reduces, and that this is then

$$[F_{ab}(x), F^{cd}(x')] = 4i(2\pi)^3 \delta_{[a}^{[c} \partial_{b]} \partial^{d]} \Delta(x - x') \quad (2.14)$$

which in momentum space is

$$[F_{ab}(p), F^{cd}(q)] = 4\delta_{[a}^{[c} p_{b]} p^{d]} \delta(p + q) \delta(p^2 - \mu^2) \epsilon(p_0). \quad (2.15)$$

Again, this is fixed by the axioms apart from the normalisation, which is chosen for later convenience. Note that this axiomatic approach bypasses the canonical quantisation procedure.

2.2. THE INTERACTING THEORY

To make the transition to interacting field theory, we assume that the interaction is characterised by a coupling constant e , that we may write the fields as a Maclaurin expansion in this parameter, and that the zeroth order terms in the expansion are free fields. Thus

$$\begin{aligned} \psi &= \psi_0 + e\psi_1 + e^2\psi_2 + \dots \\ F_{ab} &= F_{ab}^0 + eF_{ab}^1 + e^2F_{ab}^2 + \dots \end{aligned} \quad (2.16)$$

Axiom VI places constraints on the possible form of the higher order fields, and these may be solved. They are best examined in momentum space. If a pair of fields commute or anticommute for spacelike intervals, then we have

$$[\phi(\mathbf{x}, t), \chi(\mathbf{x}', t)]_\pm = C(\partial/\partial\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}') \quad (2.17)$$

where C is some finite-order polynomial in $\partial/\partial\mathbf{x}$. Thus, in terms of the Fourier transform fields,

$$\int dp^0 dp'^0 e^{i(p^0 + p'^0)t} [\phi(p), \chi(p')]_\pm = (2\pi)^{-3} C(-i\mathbf{p}) \delta(\mathbf{p} + \mathbf{p}') \quad (2.18)$$

which can be rearranged as

$$\int_{-\infty}^{\infty} d\nu [\phi(r + \nu n), \chi(q - r - \nu n)]_{\pm} = (2\pi)^{-3} C(r, n) \delta(q) \quad (2.19)$$

where $n^a = (1, 0, 0, 0)$ and $r \cdot n = 0$. Otherwise the four-vectors q and r are arbitrary. $C(r, n)$ is a quantity with the same tensor structure as the LHS, containing powers of r up to finite order. The choice of spacelike hyperplane should not be relevant, and so this must hold for all n satisfying $n_0 > 0, n^2 = 1$. We may deduce that the free fields obey

$$\begin{aligned} \int_{-\infty}^{\infty} d\nu \{ \psi_{\alpha}(r + \nu n), \bar{\psi}^{\beta}(-q + r + \nu n) \} &= (\not{n})_{\alpha}^{\beta} \delta(q) \\ \int_{-\infty}^{\infty} d\nu \{ \psi_{\alpha}(r + \nu n), \psi_{\beta}(q - r - \nu n) \} &= 0 \\ \int_{-\infty}^{\infty} d\nu [\psi_{\alpha}(r + \nu n), F_{ab}(q - r - \nu n)] &= 0 \\ \int_{-\infty}^{\infty} d\nu [F_{ab}(r + \nu n), F^{cd}(q - r - \nu n)] &= 4(\delta_{[a}^{[c} r_{b]}^{d]} n_{b]} + \delta_{[a}^{[c} n_{b]}^{d]} r_{b]} \delta(q) \end{aligned} \quad (2.20)$$

which are of course all of the right form.

If we now consider F and ψ as *interacting* fields then the modification to these is that they may develop e -dependent c -number parts on the RHS which are of the form (2.19). In fact, we find that if the RHS is a function of e , then this dependence can be eliminated by rescaling the fields by the appropriate e -dependent amount, so we need only consider the case where the presence of interactions does not affect the RHS at all.

Thus, the first-order contributions to the fields are obtained by solving the four equations

$$\begin{aligned} \int_{-\infty}^{\infty} d\nu (\{ \psi_{\alpha}^0(r + \nu n), \bar{\psi}_1^{\beta}(-q + r + \nu n) \} + \{ \psi_{\alpha}^1(r + \nu n), \bar{\psi}_0^{\beta}(-q + r + \nu n) \}) &= 0 \\ \int_{-\infty}^{\infty} d\nu (\{ \psi_{\alpha}^0(r + \nu n), \psi_{\beta}^1(q - r - \nu n) \} + \{ \psi_{\alpha}^1(r + \nu n), \psi_{\beta}^0(q - r - \nu n) \}) &= 0 \\ \int_{-\infty}^{\infty} d\nu ([\psi_{\alpha}^0(r + \nu n), F_{ab}^1(q - r - \nu n)] + [\psi_{\alpha}^1(r + \nu n), F_{ab}^0(q - r - \nu n)]) &= 0 \\ \int_{-\infty}^{\infty} d\nu ([F_{ab}^0(r + \nu n), F_{cd}^1(q - r - \nu n)] + [F_{ab}^1(r + \nu n), F_{cd}^0(q - r - \nu n)]) &= 0 \end{aligned} \quad (2.21)$$

for ψ_1 and F_{ab}^1 . This can be done fairly straightforwardly. Consider the first equation. It follows from this that

$$\begin{aligned} \int_{-\infty}^{\infty} d\nu (\not{n}(\not{r} + \nu \not{n}) \{ \psi^1(r + \nu n), \bar{\psi}^0(r + \nu n - q) \} \\ - \{ \psi^0(r + \nu n), \bar{\psi}^1(r + \nu n - q) \} (\not{r} + \nu \not{n} - \not{q} + m) \not{n}) = 0 \end{aligned} \quad (2.22)$$

since the $\{ \psi^0, \bar{\psi}^1 \}$ term in the first expression is projected out, as is the $\{ \psi^1, \bar{\psi}^0 \}$ term in the second; and the order ν parts cancel. If we define the functional derivative thus:

$$\frac{\delta S[\bar{\psi}_0]}{\delta \bar{\psi}_0^{\alpha}(q)} = \lim_{\epsilon \rightarrow 0} \epsilon^{-1} (S[\bar{\psi}_0^{\gamma}(q') + \epsilon \delta(q - q') \delta_{\alpha}^{\gamma}] - S[\bar{\psi}_0^{\gamma}(q')]) \quad (2.23)$$

where ϵ is a parameter which anticommutes with each Fermi field, then it follows that

$$\begin{aligned} \{ \psi_{\alpha}^0(p), S[\bar{\psi}_0] \} &= (\not{q} - m)_{\alpha}^{\beta} \frac{\delta S[\bar{\psi}_0]}{\delta \bar{\psi}_0^{\beta}(p)} \delta(q^2 - m^2) \epsilon(q_0) \\ \text{and } \{ S[\psi_0], \bar{\psi}_0^{\beta}(q) \} &= \frac{\delta S[\psi_0]}{\delta \psi_0^{\alpha}(q)} (\not{q} - m)_{\alpha}^{\beta} \delta(q^2 - m^2) \epsilon(q_0). \end{aligned} \quad (2.24)$$

So the problem is to solve

$$\int d\nu \left\{ \not{n} \frac{\delta \tilde{\psi}_1(r + \nu n)}{\delta \psi_0(r + \nu n - q)} (\not{r} + \nu \not{n} - \not{q} - m) \delta((r + \nu n - q)^2 - m^2) \epsilon(r_0 + \nu n_0 - q_0) \right. \\ \left. - (\not{r} + \nu \not{n} - m) \frac{\delta \bar{\psi}_1(r + \nu n - q)}{\delta \bar{\psi}_0(r + \nu n)} \delta((r + \nu n)^2 - m^2) \epsilon(r_0 + \nu n_0) \right\} = 0 \quad (2.25)$$

$$\Rightarrow \frac{1}{2S} \left[\not{n} \frac{\delta \tilde{\psi}_1(q - s_-)}{\delta \psi_0(-s_-)} (-\not{s}_- - m) - \not{n} \frac{\delta \tilde{\psi}_1(q - s_+)}{\delta \psi_0(-s_+)} (-\not{s}_+ - m) \right] \\ - \frac{1}{2R} \left[(\not{r}_+ - m) \frac{\bar{\psi}_1(r_+ - q)}{\delta \bar{\psi}_0(r_+)} \not{n} - (\not{r}_- - m) \frac{\delta \bar{\psi}_1(r_- - q)}{\delta \bar{\psi}_0(r_-)} \not{n} \right] = 0 \quad (2.26)$$

where we use the notation $\tilde{\psi}(q) = (\not{q} + m)\psi(q)$, $R = \sqrt{(m^2 - r^2)}$, $s = q - r - (n \cdot q)n$, $S = \sqrt{(m^2 - s^2)}$, $r_{\pm} = r \pm Rn$ and $s_{\pm} = s \pm Sn$. We now apply $(\not{r}_{\mp} + m)$ to the left, and $(\not{s}_{[\mp]} - m)$ to the right (the square brackets mean an independent choice of signs). This leads to

$$(\not{r}_{\mp} + m) \not{n} \left[\frac{\delta \tilde{\psi}_1(q - s_{[\pm]})}{\delta \psi_0(-s_{[\pm]})} - \frac{\delta \bar{\psi}_1(r_{\pm} - q)}{\delta \bar{\psi}_0(r_{\pm})} \right] \not{n} (\not{s}_{[\mp]} - m) = 0 \quad (2.27)$$

for which the solution is

$$\frac{\delta \psi_{\alpha}^1(p)}{\delta \psi_{\beta}^0(q)} = [(\not{p} + m)^{-1} C(p - q)]_{\alpha}^{\beta} \quad (2.28)$$

where C is an operator-valued quantity satisfying

$$C(q) = \gamma^0 C^+(-q) \gamma^0. \quad (2.29)$$

The solution of the $\{\psi, \psi\}$ anticommutator proceeds in the same way. This leads to

$$\frac{\delta \psi_{\alpha}^1(p)}{\delta \bar{\psi}_{\beta}^0(-q)} = (\not{p} + m)^{-1}_{\alpha}{}^{\gamma} N_{\gamma\beta}(p - q) \quad (2.30)$$

where N is an operator-valued quantity satisfying

$$N_{\gamma\beta}(q) = -N_{\beta\gamma}(q). \quad (2.31)$$

The $[\psi, F]$ commutator leads to

$$\frac{\delta \tilde{A}_a^1(q - r_{\pm})}{\delta \bar{\psi}_0(r_{\pm})} - (r_{\pm} + m) \alpha_a(q, r_{\pm}) - \beta(q, r_{\pm})(q - r_{\pm})_a = \frac{\delta \tilde{\psi}_1(q - s_{[\pm]})}{\delta A_0^a(-s_{[\pm]})} + \gamma(q, s_{[\pm]}) \cdot s_a^{[\pm]} = V_a(q) \quad (2.32)$$

where $S = \sqrt{(\mu^2 - s^2)}$ now, with μ the photon mass, and α, β, γ and V arbitrary operator-valued functions. We use the notation $\tilde{A}_a(q) = (q^2 - \mu^2)A_a(q)$, and the functional derivative with respect to A is analogous to (2.23) except that ϵ is a normal scalar parameter. The fact that this appears only in the functional derivative $\delta/\delta F_0$ means that it has the property that

$$p^a \frac{\delta(\dots)}{\delta A_0^a(p)} = 0. \quad (2.33)$$

Thus, contracting (2.32) with $s_{[\pm]}$, we find that

$$\mu^2 \cdot \gamma(q, s_{[\pm]}) = s_{[\pm]} \cdot V(q). \quad (2.34)$$

So if $\mu = 0$, the interaction $V(q)$ is trivial. We conclude from this that in this theory *the photon may not be completely massless*. Accepting this limitation, the commutator leads to

$$\begin{aligned} \frac{\delta\psi_{1\alpha}(p)}{\delta A_0^a(q)} &= (\not{p} + m)^{-1} \alpha^\beta (V_{a\beta}(p - q) - \frac{q_a q^b}{\mu^2} V_{b\beta}(p - q)) \\ \text{and } \frac{\delta A_a^1(p)}{\delta \bar{\psi}_0^\alpha(-q)} &= (p^2 - \mu^2)^{-1} [V_{a\alpha}(p - q) + \beta(p, q) p_a] \end{aligned} \quad (2.35)$$

The final commutator $[F, F]$ leads to

$$\frac{\delta A_a^1(p)}{\delta A_0^b(q)} = (p^2 - \mu^2)^{-1} [f_{ab}(p - q) - \frac{q_b q^c}{\mu^2} f_{ac}(p - q) + p_a (\delta_b(p, q) - \frac{q_b q^c}{\mu^2} \delta_c(p, q))] \quad (2.36)$$

where $f_{ab}(q) = f_{ab}^+(-q) = f_{ba}(q)$ and $\delta_a^+(p, q) = -\delta_a(-p, -q)$ but these are otherwise arbitrary.

The solution to these constraints can be obtained by writing out the most general forms for ψ_1 and A_1 in terms of ψ_0 and A_0 , and then applying functional derivatives. We find that the solutions are

$$\psi_1(p) = (\not{p} + m)^{-1} \frac{\delta S_0(q)}{\delta \bar{\psi}_0(p - q)} \quad \text{and} \quad A_{1a}(p) = (p^2 - \mu^2)^{-1} \left[\frac{\delta S_0(q)}{\delta A_0^a(q - p)} + p_a B(p) \right] \quad (2.37)$$

where $B(p) = -B^+(-p)$ is related to β by $\beta = \delta B / \delta \bar{\psi}$; and $S_0(q)$ is a local, non-derivative, Lorentz-invariant construction of ψ_0 , $\bar{\psi}_0$ and A_0 such that

$$S_0(q)^+ = S_0(-q); \quad (2.38)$$

i.e. it has the form

$$\begin{aligned} S_0(q) &= \sum_{\substack{\text{various} \\ n, m}} M_{\alpha_1 \dots \alpha_n}{}^{a_1 \dots a_m \beta_1 \dots \beta_n} \int d^4 p_1 \dots d^4 p_n d^4 q_1 \dots d^4 q_m d^4 r_1 \dots d^4 r_n \\ &\quad \delta(q - p_1 - \dots - p_n - q_1 - \dots - q_m - r_1 - \dots - r_n) \\ &\quad \bar{\psi}_0^{\alpha_1}(-p_1) \dots \bar{\psi}_0^{\alpha_n}(-p_n) A_{a_1}^0(q_1) \dots A_{a_m}^0(q_m) \psi_{\beta_1}(r_1) \dots \psi_{\beta_n}(r_n) \end{aligned} \quad (2.39)$$

where M is a preserved tensor of $SL(2, \mathcal{C})$, which does not violate parity, and satisfies

$$M_{\alpha_1 \dots \alpha_n}{}^{a_1 \dots a_m \beta_1 \dots \beta_n} = M_{\beta'_1 \dots \beta'_m}{}^{a'_1 \dots a'_m \alpha'_1 \dots \alpha'_n} (\gamma^0)^{\beta_1 \beta'_1} \dots (\gamma^0)^{\beta_m \beta'_m} (\gamma^0)_{\alpha_1 \alpha'_1} \dots (\gamma^0)_{\alpha_n \alpha'_n} \quad (2.40)$$

If we stipulate that A_a is a vector (rather than an axial vector), and that the coupling is trilinear, then we have

$$S_0(q) = - \int d^4 p_1 d^4 q_1 d^4 r_1 \delta(q - p_1 - q_1 - r_1) \bar{\psi}_0(-p_1) A_0(q_1) \psi_0(r_1). \quad (2.41)$$

which is unique (apart from a scaling factor, which may in any case be absorbed by redefining e); which gives us

$$\begin{aligned} (\not{p} + m)\psi_1(p) &= - \int d^4 q A_0(q) \psi_0(p - q) \\ (p^2 - \mu^2)A_1^a(p) &= - \int d^4 q \bar{\psi}_0(q) \gamma^a \psi_0(p + q) + p^a B(p) \end{aligned} \quad (2.42)$$

The higher-order terms in the expansion can now be derived straightforwardly. For these, we need the commutators

$$\begin{aligned} &\int_{-\infty}^{\infty} d\nu [\psi(r + \nu n), A_a(q - r - \nu n)]_1 \\ \text{and } &\int_{-\infty}^{\infty} d\nu \nu [\psi(r + \nu n), A_a(q - r - \nu n)]_1 \end{aligned} \quad (2.43)$$

where we use the notation

$$(\phi\chi\psi\cdots)_n = \sum_{i,j,k,\cdots} \delta_{n,i+j+k+\cdots} \phi_i\chi_j\psi_k\cdots \quad (2.44)$$

Substituting the solutions obtained earlier [2.35], we have

$$\begin{aligned} \int_{-\infty}^{\infty} d\nu [\psi(r+\nu n), A_a(q-r-\nu n)]_1 &= -\frac{1}{\mu^2} \not{n}_a (n \cdot V(q)) \\ &+ \frac{1}{2R} [(\not{r}_+ - m)((\xi - R)^2 - S^2)^{-1} \{\beta_+ + \frac{(q-r_+) \cdot V}{\mu^2}\} (q-r_+)_a \\ &- (\not{r}_- - m)((\xi + R)^2 - S^2)^{-1} \{\beta_- + \frac{(q-r_-) \cdot V}{\mu^2}\} (q-r_-)_a] \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} d\nu \nu [\psi(r+\nu n), A_a(q-r-\nu n)]_1 &= \frac{1}{\mu^2} (\not{s}_a (n \cdot V) + n_a (\not{s} \cdot V) - (\not{r} - m)(n \cdot V)) \\ &+ \frac{1}{2} \left[(\not{r}_+ - m) \frac{\beta_+ + \frac{(q-r_+) \cdot V}{\mu^2}}{(\xi - R)^2 - S^2} (q-r_+)_a + (\not{r} - m) \frac{\beta_- + \frac{(q-r_-) \cdot V}{\mu^2}}{(\xi + R)^2 - S^2} (q-r_-)_a \right] \end{aligned} \quad (2.45)$$

where $\beta_{\pm} = \beta(q-r_{\pm}, -r_{\pm})$ of eq.(2.35) and $\xi = n \cdot q$.

Evidently, if $\beta_{\pm} = -\mu^{-2}(q-r_{\pm}) \cdot V(q)$, then the expressions take a particularly simple form, i.e.

$$\begin{aligned} \int_{-\infty}^{\infty} d\nu [\psi(r+\nu n), A_a(q-r-\nu n)]_1 &= -\frac{\not{n}_a (n \cdot V)}{\mu^2} \\ \int_{-\infty}^{\infty} d\nu \nu [\psi(r+\nu n), A_a(q-r-\nu n)]_1 &= \frac{\not{s}_a (n \cdot V) + n_a (\not{s} \cdot V) - (\not{r} - m)(n \cdot V)}{\mu^2} \end{aligned} \quad (2.46)$$

Choosing the solution with S_0 given by (2.41), we find that the first-order fields are then given by

$$\begin{aligned} (\not{p} + m)\psi_1(p) &= -\int d^4q \mathcal{A}_0(q)\psi_0(p-q) \\ (p^2 - \mu^2)A_1^a(p) &= -\int d^4q \bar{\psi}_0(q) \left(\gamma^a - \frac{p^a}{\mu^2} \not{p}\right) \psi_0(p+q) \\ &= -\int d^4q \bar{\psi}_0(q) \gamma^a \psi_0(p+q) \end{aligned} \quad (2.47)$$

We may now show that

$$\begin{aligned} \tilde{\psi}_{n+1}(p) &= -\int d^4q (\mathcal{A}(q)\psi(p-q))_n \\ \text{and } \tilde{A}_a^{n+1}(p) &= -\int d^4q (\bar{\psi}(q)\gamma_a\psi(p+q))_n \end{aligned} \quad (2.48)$$

$$\begin{aligned} \text{with } \int_{-\infty}^{\infty} d\nu [\psi(r+\nu n), A_a(q-r-\nu n)]_{n+1} &= \frac{1}{\mu^2} \psi_n(q) n_a \\ \int_{-\infty}^{\infty} d\nu \nu [\psi(r+\nu n), A_a(q-r-\nu n)]_{n+1} &= -\frac{(s-\xi)_a}{\mu^2} \psi_n(q) \\ \int_{-\infty}^{\infty} d\nu \{1, \nu, \nu^2\} [A_a(r+\nu n), A_b(q-r-\nu n)]_{n+1} &= 0 \end{aligned} \quad (2.49)$$

will solve the (anti)commutators to all orders. That the forms (2.49) satisfy the $[\psi, F]$ and $[F, F]$ commutators (i.e. make them vanish) is easy to deduce, as is the fact that they are equivalent to (2.46) for $n = 0$. For $n > 0$, we substitute the solutions (2.48) into (2.49) or the anticommutators $\{\psi, \bar{\psi}\}_{n+1}$ and $\{\psi, \psi\}_{n+1}$ being required to be zero, and thereby check the consistency.

For example, verifying

$$\int d\nu \{\psi_\alpha(r + \nu n), \bar{\psi}^\beta(r + \nu n - q)\}_{n+1} = 0;$$

multiply by $\not{n}(\not{r} + m) \rightarrow -\not{r}(\not{r} - \not{q} + m) \not{n}$, so that we have

$$\int d\nu (\not{n}(\not{r} + \nu \not{n} + m)\{\psi(r + \nu n), \bar{\psi}(r + \nu n - q)\}_{n+1} - \{\psi(r + \nu n), \bar{\psi}(r + \nu n - q)\}_{n+1} (\not{r} + \nu \not{n} - \not{q} + m) \not{n}) = 0 \quad (2.50)$$

which is

$$\int d\nu (\not{n}\{\tilde{\psi}(r + \nu n), \bar{\psi}(r + \nu n - q)\} - \{\psi(r + \nu n), \bar{\psi}(r + \nu n - q)\}\not{n})_{n+1} = 0. \quad (2.51)$$

Replacing the tilde fields with the expansions, we then get

$$- \int d^4 q' \int d\nu (\not{n}\{\not{A}(q')\psi(r + \nu n - q'), \bar{\psi}(r + \nu n - q)\} - \{\psi(r + \nu n), \bar{\psi}(-q + r + \nu n + q') \not{A}(q')\})_n = 0 \quad (2.52)$$

Expanding the anticommutators with

$$\{AB, C\} = A\{B, C\} - [A, C]B, \quad (2.53)$$

and substituting the lower-order expressions for these terms, we see that the equation holds. To ensure that the reverse implication works for commutators involving A , we have to check different combinations—e.g. to establish

$$\int d\nu (1, \nu, \nu^2)[A, A]_{n+1} = 0 \quad (2.54)$$

we have to check that the independent combinations

$$\int d\nu [\tilde{A}, A], \quad \int d\nu [A, \tilde{A}] \quad \text{and} \quad \int d\nu \nu ([A, \tilde{A}] - [\tilde{A}, A]) \quad (2.55)$$

are all zero.

We have therefore shown that the equations of motion of normal massive quantum electrodynamics

$$\begin{aligned} (\not{p} + m)\psi(p) &= -e \int d^4 q \not{A}(q)\psi(p - q) \\ (p^2 - \mu^2)A^a(p) &= -e \int d^4 q \bar{\psi}(q)\gamma^a\psi(p + q) \end{aligned} \quad (2.56)$$

which are

$$\begin{aligned} (i\not{\partial} - e \not{A} - m)\psi &= 0 \\ (\partial^2 + \mu^2)A_a &= e\bar{\psi}\gamma_a\psi \end{aligned} \quad (2.57)$$

in configuration space (together with the stipulation $\partial \cdot A = 0$), are the simplest interacting solution to the axioms presented at the beginning of this section. This solution is not unique, though. It appears that any local, non-derivative, Lorentz-invariant, parity-conserving construction derived appropriately through the Action principle, will solve the (anti)commutators (although this has not been checked fully).

We may always reduce the interacting field to combinations of free fields, as we have seen. Since the properties of these are well established, it follows that any Green function can be written down by inspection. To see this, we note that (e.g.)

$$[P_a, \psi_0(x)] = -i\partial_a \psi_0(x) \quad (2.58)$$

remains true even in the presence of interactions, on account of the fact that P_a is just the translation generator. Hence

$$[P_a, \psi_0(p)] = p_a \psi_0(p) \quad (2.59)$$

and so

$$\psi_0(p)|0\rangle \quad (2.60)$$

is a state of four-momentum p_a . But axiom *IV* requires that there are no negative energy states, so this must just give zero when $p_0 < 0$.

The procedure for obtaining the value of a Green function is thus: write out the expansion of each interacting field in terms of free fields; then commute or anticommute the negative-energy parts of these to the right to annihilate the vacuum. The value of the Green function is then just the value of the (anti)commutator c-numbers picked up in the process.

The higher-order fields reduce to lower-order ones through

$$\begin{aligned} \psi_n(p) &= -(\not{p} + m)^{-1} \int d^4q \sum_i A_i(q) \psi_{n-1-i}(p-q) \\ A_n^a(p) &= -(p^2 - \mu^2)^{-1} \int d^4q \sum_i \bar{\psi}_i(q) \gamma^a \psi_{n-1-i}(p+q). \end{aligned} \quad (2.61)$$



Fig. 1. Graphical representation of the first-order fields

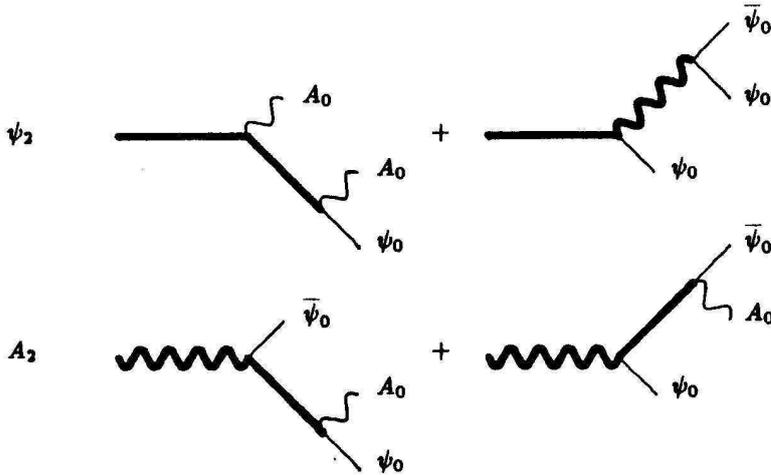


Fig. 2. Graphical representation of the second-order fields

These can be viewed graphically as trees whose branches have free fields attached at the ends. For example, we can see ψ_1 and A_1 as the stems of fig. 1. We associate the factors $-(\not{p}+m)^{-1}$ and $-(p^2-\mu^2)^{-1}$ with the heavy lines, called “proliferators”. The second-order fields then have the representations shown in fig. 2, since we now attach one zeroth order and one first order field to the proliferator in each case. The higher order fields we obtain by continuing the process, i.e. adding further branches to the trees. In a vacuum expectation value of a product of fields, the trees which represent each field link up through the (anti)commutation of the free fields to the right to annihilate the vacuum. This leads to the following set of rules for obtaining the value of a Green function

2.3. THE GRAPH RULES

(1). The matrix element is given by the sum of expressions corresponding to all of the topologically possible diagrams, subject to the following rules. There are no disconnected diagrams. For each field in the matrix element corresponding to an external particle we assign a momentum such that the time component will be positive. For incoming particles we associate a line which enters the diagram from the right. The incoming lines appear from top to bottom in the same order that the fields in the matrix element appear from right to left. Outgoing particles are represented by lines leaving the diagram to the left, and the order that they appear from top to bottom is the same order as they appear in the matrix element from left to right (see fig. 3).

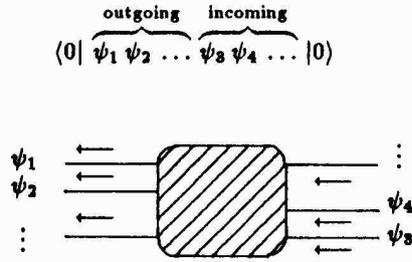


Fig.3. Graphical representation of a matrix element

(2). There are two kinds of line:

i. A proliferator. A heavy line.

(a). Fermions. The proliferators have arrows to indicate the direction of charge flow. This follows momentum flow for electrons and opposes it for positrons. There is a factor

$$\begin{aligned} &(\not{p} - m)^{-1} \quad \text{for an electron, and} \\ &-(\not{p} + m)^{-1} \quad \text{for a positron.} \end{aligned}$$

These are represented by straight lines.

(b). Photons. Wavy line with a factor $-\eta_{ab}(p^2 - \mu^2)^{-1}$.

From every proliferator it must be possible to access a unique external line by a unique route made up solely of proliferators (i.e. there are no (just-) proliferator loops, and we cannot go from one external line to another by proliferators alone). Momentum flow is always outwards along the tree for incoming lines and inwards (i.e. towards the stem) for outgoing lines.

ii. A propagator. A thin line.

(a). Fermions.

$$\begin{aligned} &(\not{p} + m) \theta(p_0) \delta(p^2 - m^2) \quad \text{for an electron, and} \\ &(\not{p} - m) \theta(p_0) \delta(p^2 - m^2) \quad \text{for a positron.} \end{aligned}$$

A straight line. We draw an arrow next to the line to show momentum flow, and an arrow on the line to show charge flow—which follows momentum flow for electrons, and opposes it for positrons.

(b). Photons. Wavy line with a factor

$$(-\eta_{ab} + \frac{p_a p_b}{\mu^2}) \theta(p_0) \delta(p^2 - \mu^2).$$

We draw an arrow next to the line to show momentum flow.

The direction of momentum flow in propagators is always from incoming to outgoing lines, or downwards if between incoming proliferator trees, or upwards if between outgoing proliferator trees. Charge flow in the proliferator trees is *downwards* in incoming trees. Propagators may attach at both ends to the same proliferator tree. In this case we have to arrange that the momentum flow of the propagators is then downwards. These directions become upwards for outgoing trees. Charge flow in loops is always *clockwise*. No line may cross a proliferator.

(3). The vertices are always the junctions of two fermion and one photon line. Four-momentum and charge are conserved here, and a factor $e\gamma_a$ is associated with it. There must be at least one proliferator joining the vertex.

(4). Spinor matrices are always contracted in the reverse order as they appear along a charge line. There is a factor (-1) for each crossing of fermion line by fermion line. For each graph, or connected subgraph, there is a momentum conservation factor $\delta(\sum p_f - \sum p_i)$. Undetermined momenta are integrated over (but without an attached numerical factor).

2.4. THE INFINITIES AND THEIR REMOVAL

We obtain infinite answers for all physical quantities on account of higher order graphs. This means that the theory we have been developing cannot be the final answer. One hope is that these infinities are a consequence of the expansion in the coupling constant, and if this could be avoided then they would disappear, but this is just speculation. For the time being we will “learn to live” with these since, if we do, we find that we reproduce Feynman graphs, at least, up to “tree” level, and these are known to give good agreement with experiment.

If each field in the matrix element could be written as a normal-ordered reduction into free fields then no infinities would appear at all. The infinities arise *within* the power series reduction of each field, and graphically will appear as propagators attached at both ends to the same proliferator tree. The types, and their treatment, are as follows:

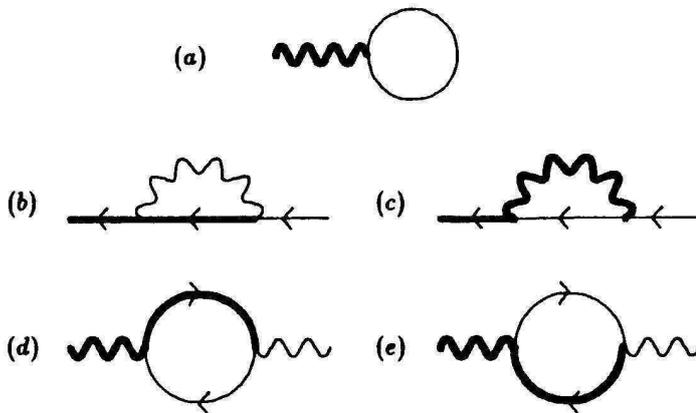


Fig. 4. Infinite graphs up to second order

(i) Order e , graphically represented by the tadpole graph (fig. 4a). This is an infinite scalar which we obtain when putting $\bar{\psi}_0\psi_0$ into normal order in A_a^1 . This presents no difficulty at all since the normal-ordered coupling is still strictly a solution of the commutators [2.21]—as may be checked by functional differentiation.

(ii) Order e^2 . Figs. 4b and c are the infinities associated with ψ_2 ; and figs. 4d and e are the ones associated with A_2 . In obtaining expressions for the higher-order fields we did not actually solve the commutators directly—we merely substituted expressions, and showed that these worked. However, it is possible to see how the direct method would apply. To obtain ψ_2 and F_2 we would solve

$$\int d\nu (\{\psi_0, \bar{\psi}_2\} + \{\psi_1, \bar{\psi}_1\} + \{\psi_2, \bar{\psi}_0\}) = 0 \quad (2.62)$$

together with $\{\psi, \psi\}_2$, $[\psi, F]_2$ and $[F, F]_2$ being required to vanish. The middle term of (2.62) takes the form of a particular integral, and in analogy to the solution of the first-order case, we may derive

$$\begin{aligned}
(\not{r}'_{\mp} + m) \not{r} \left[\frac{\delta}{\delta\psi_0(-s_{[\pm]})} \left\{ \tilde{\psi}_2(q - s_{[\pm]}) + \int d^4q' (\mathcal{A}(q')\psi(q - s_{[\pm]} - q'))_1 \right\} \right. \\
\left. - \frac{\delta}{\delta\bar{\psi}_0(r_{\pm})} \left\{ \bar{\psi}_2(r_{\pm} - q) + \int d^4q' (\bar{\psi}(q') \mathcal{A}(q - r_{\pm} + q'))_1 \right\} \right] \not{r} (\not{s}'_{[\mp]} - m) = 0.
\end{aligned} \tag{2.63}$$

The second-order solution—as any higher order solution—is arbitrary to the extent of addition of a first-order solution. If we use the freedom available to include a first-order solution, then we are essentially modifying the equation of motion (such as (2.56)) to include local couplings of order e^2 and higher. If we do *not* allow this, then we are making each functional derivative in (2.63) separately zero, and may then integrate to obtain the expression for ψ_2 that we had before [2.48]. However, these give the aforementioned infinities, which may only be avoided if we put the interactions $(\mathcal{A}\psi)_1$ and $(\bar{\psi}\mathcal{A})_1$ in (2.63), in normal order. We must do this, so we have to subtract infinity times $\psi_0(q - s_{[\pm]})$ from the first interaction, and infinity times $\bar{\psi}_0(r_{\pm} - q)$ in the other interaction. The tensors outside the main bracket have no effect, so what we are now attempting to solve is thus

$$\begin{aligned}
\frac{\delta}{\delta\psi_0(-s_{[\pm]})} \left\{ \tilde{\psi}_2(q - s_{[\pm]}) + \int d^4q' : (\mathcal{A}(q')\psi(q - s_{[\pm]} - q'))_1 : \right\} \\
- \frac{\delta}{\delta\bar{\psi}_0(r_{\pm})} \left\{ \bar{\psi}_2(r_{\pm} - q) + \int d^4q' : (\bar{\psi}(q') \mathcal{A}(q - r_{\pm} + q'))_1 : \right\} \\
= \text{indeterminate scalar}
\end{aligned} \tag{2.64}$$

since $(\infty) - (\infty)$ is indeterminate.

The LHS is of the form $f(q, s_{[\pm]}) - \bar{f}(-q, -r_{\pm})$, so either the theory is inconsistent, or the scalar takes the form $s(q, s_{[\pm]}) - s^*(-q, -r_{\pm})$ and we are presented with the problem of solving

$$\frac{\delta}{\delta\psi_0(-s_{[\pm]})} \left\{ \tilde{\psi}_2(q - s_{[\pm]}) + \int d^4q' : (\mathcal{A}(q')\psi(q - s_{[\pm]} - q'))_1 : \right\} - s(q, s_{[\pm]}) = C_2(q). \tag{2.65}$$

We take $C_2(q)$ to be zero for reasons given earlier. The scalar must then be zero as will be seen when we put the resultant $\tilde{\psi}_2(p)$ on-shell and take vacuum expectation values. The same procedure may be applied to the $[F, F]_2$ commutator to normal-order \tilde{A}_2 , and the commutators $[\psi, \psi]_2$ and $[\psi, F]_2$ will still be zero. We may, therefore, ignore the divergent graphs.

(iii) Order e^3 and higher. The foregoing may be extended to higher-order infinities, except that instead of getting indeterminate scalars, we will normally get indeterminate operators, which, unlike the scalars obtained before can in fact be legitimately integrated to make a contribution to the higher-order field. The indeterminacy means that this extra contribution is arbitrary except in the number and types of each field that it involves—so it may be a derivative or non-local construction, and as long as these appear in normal order, we cannot to exclude such contributions with complete rigour. We will exclude these terms on the grounds that this is not the theory being considered: which is our argument for excluding higher than trilinear couplings—which would be allowed from the solution of the commutators alone. Fundamentally, the problem is that locality, as expressed by axiom VI is something that cannot unambiguously be demanded of an interacting field theory. Our theory then represents what is apparently the simplest attempt at consistency with (VI) that is available in an interacting theory.

The consequence of all this, then, is that we delete all the infinite graphs which appear—which will always be ones where propagators are attached at both ends to the same proliferator tree (incidentally, finite graphs

of this kind must contain at least three photon proliferators in each loop, so they cannot appear until we are solving for ψ_6 or A_7).

3. Scattering processes

We will consider the term “scattering” process to mean one where the particles involved behave as though free for almost all of the time, interacting only briefly within some well-defined spatial region. In such a situation the amplitudes are well approximated by the lowest-order contributions to the matrix elements. We will consider the specific cases of e^+e^- elastic scattering and $e^+e^- \rightarrow \gamma\gamma\gamma$, and then go on to the general case, showing that the lowest-order contributions to such processes are the same as those obtained from the tree-level Feynman graphs.

3.1. e^+e^- ELASTIC SCATTERING

The Green function we need is

$$\langle 0 | \psi_\alpha(t; -\mathbf{q}_1) \bar{\psi}^\beta(t; \mathbf{q}_2) \psi_\gamma(0; \mathbf{p}_2) \bar{\psi}^\delta(0; -\mathbf{p}_1) | 0 \rangle = \int dq_1^0 dq_2^0 dp_1^0 dp_2^0 e^{-i(q_1^0 + q_2^0)t} \langle 0 | \psi_\alpha(-q_1) \bar{\psi}^\beta(q_2) \psi_\gamma(p_2) \bar{\psi}^\delta(-p_1) | 0 \rangle \quad (3.1)$$

The matrix element $\langle 0 | \psi_\alpha(-q_1) \bar{\psi}^\beta(q_2) \psi_\gamma(p_2) \bar{\psi}^\delta(-p_1) | 0 \rangle$ is, up to order e^2 , the sum of the following terms, represented by the graphs of fig. 5:

$$(\not{p}_1 + m)_\alpha^\delta \theta(p_1^0) \delta(p_1^2 - m^2) \delta(p_1 - q_1) (\not{p}_2 - m)_{\gamma^\beta} \theta(p_2^0) \delta(p_2^2 - m^2) \delta(p_2 - q_2) \quad (3.2a)$$

$$e^2 \left[(\not{p}_1 - m)^{-1} \int d^4q \theta(q^0) \left(-\eta^{ab} + \frac{q^a q^b}{\mu^2} \right) \delta(q^2 - \mu^2) \gamma^b \theta(p_1^0 - q^0) \delta((p_1 - q)^2 - m^2) \right. \\ \left. (\not{p}_1 - \not{q} + m) \gamma^a (\not{p}_1 - m)^{-1} \right]_\alpha^\delta \delta(p_1 - q_1) \\ (\not{p}_2 - m)_{\gamma^\beta} \theta(p_2^0) \delta(p_2^2 - m^2) \delta(p_2 - q_2) \quad (3.2b)$$

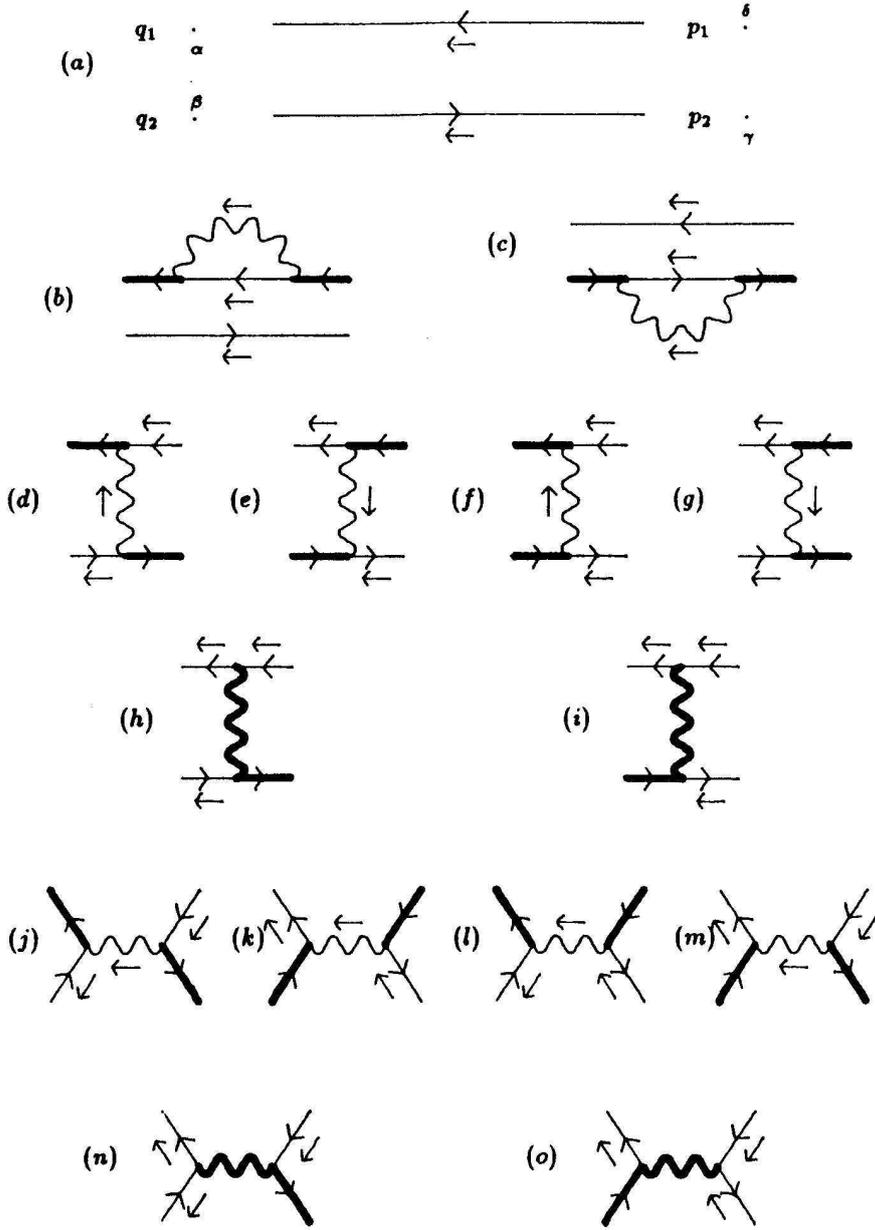


Fig. 5. The contributions to e^+e^- elastic scattering up to order e^2

$$\begin{aligned}
& e^2 (\not{p}_1 + m)_\alpha^\delta \theta(p_1^0) \delta(p_1^2 - m^2) \delta(p_1 - q_1) \left[(\not{p}_2 + m)^{-1} \int d^4q \theta(q^0) \delta(q^2 - \mu^2) \right. \\
& \quad \left. \left(-\eta^{ab} + \frac{q^a q^b}{\mu^2} \right) \gamma^b \theta(p_2^0 - q^0) \delta((p_2 - q)^2 - m^2) (\not{p}_2 - \not{q} - m) \right. \\
& \quad \left. \gamma^a (\not{p}_2 + m)^{-1} \right]_\gamma^\beta \delta(p_2 - q_2) \quad (3.2c) \\
& e^2 [(\not{q}_1 - m)^{-1} \gamma^a (\not{p}_1 + m)]_\alpha^\delta \theta(p_1^0) \delta(p_1^2 - m^2) \left(-\eta_{ab} + \frac{(p_2 - q_2)_a (p_2 - q_2)_b}{\mu^2} \right)
\end{aligned}$$

$$\theta(p_2^0 - q_2^0) \delta((p_2 - q_2)^2 - \mu^2) (-1) [(\not{p}_2 + m)^{-1} \gamma^b \theta(q_2^0) \delta(q_2^2 - m^2) (\not{q}_2 - m)]_{\gamma}^{\beta} \delta(q_1 + q_2 - p_1 - p_2) \quad (3.2d)$$

plus corresponding expressions for the graphs e, f and g.

$$e^2 \theta(q_1^0) \delta(q_1^2 - m^2) [(\not{q}_1 + m) \gamma^a \theta(p_1^0) \delta(p_1^2 - m^2) (\not{p}_1 + m)]_{\alpha}^{\delta} \frac{-1}{(p_1 - q_1)^2 - \mu^2} (-1) [(\not{p}_2 + m)^{-1} \gamma_a \theta(q_2^0) \delta(q_2^2 - m^2) (\not{q}_2 - m)]_{\gamma}^{\beta} \delta(q_1 + q_2 - p_1 - p_2) \quad (3.2h)$$

$$e^2 \theta(q_1^0) \delta(q_1^2 - m^2) [(\not{q}_1 + m) \gamma^a \theta(p_1^0) \delta(p_1^2 - m^2) (\not{p}_1 + m)]_{\alpha}^{\delta} \frac{-1}{(p_1 - q_1)^2 - \mu^2} \theta(p_2^0) \delta(p_2^2 - m^2) (\not{p}_2 - m) \gamma_a (-1) (\not{q}_2 + m)^{-1}]_{\gamma}^{\beta} \delta(q_1 + q_2 - p_1 - p_2) \quad (3.2i)$$

$$-e^2 [(\not{p}_2 + m)^{-1} \gamma_a \theta(p_1^0) \delta(p_1^2 - m^2) (\not{p}_1 + m)]_{\gamma}^{\delta} \left(-\eta^{ab} + \frac{(p_1 + p_2)^a (p_1 + p_2)^b}{\mu^2} \right) \theta(p_1^0 + p_2^0) \delta((p_1 + p_2)^2 - \mu^2) [(\not{q}_1 - m)^{-1} \gamma_b (\not{q}_2 - m)]_{\alpha}^{\beta} \theta(q_2^0) \delta(q_2^2 - m^2) \delta(q_1 + q_2 - p_1 - p_2) \quad (3.2j)$$

plus corresponding expressions for graphs k, l and m.

$$e^2 [(\not{q}_1 + m) \theta(q_1^0) \delta(q_1^2 - m^2) \gamma_a (\not{q}_2 - m)]_{\alpha}^{\beta} \theta(q_2^0) \delta(q_2^2 - m^2) \frac{-1}{(p_1 + p_2)^2 - \mu^2} [-(\not{p}_2 + m)^{-1} \gamma^a \theta(p_1^0) \delta(p_1^2 - m^2) (\not{p}_1 + m)]_{\gamma}^{\delta} \delta(q_1 + q_2 - p_1 - p_2) \quad (3.2n)$$

$$e^2 [(\not{q}_1 + m) \theta(q_1^0) \delta(q_1^2 - m^2) \gamma_a (-1) (\not{q}_2 + m)^{-1}]_{\alpha}^{\beta} \frac{-1}{(p_1 + p_2)^2 - \mu^2} [(\not{p}_2 - m) \theta(p_2^0) \delta(p_2^2 - m^2) \gamma^a \theta(p_1^0) \delta(p_1^2 - m^2) (\not{p}_1 + m)]_{\gamma}^{\delta} \delta(q_1 + q_2 - p_1 - p_2) \quad (3.2o)$$

The integrals in the self-energy graphs are just phase space integrals and so do not diverge. For example, graph (b) yields

$$\frac{\pi e^2}{2p_1^2} \theta\left(p_1^0 - \sqrt{\mathbf{p}_1^2 + (m + \mu)^2}\right) \sqrt{(p_1^2 - (\mu + m)^2)(p_1^2 - (\mu - m)^2)} \left[\left(-\not{p}_1 + 3m - \frac{(p_1^2 - m^2 - 2\mu^2)(p_1^2 + \mu^2 - m^2)}{2\mu^2 p_1^2} \not{p}_1 \right) (\not{p}_1 - m)^{-1} \right]_{\alpha}^{\delta} \delta(p_1 - q_1) (\not{p}_2 - m)_{\gamma}^{\beta} \theta(p_2^0) \delta(p_2^2 - m^2) \delta(p_2 - q_2) \quad (3.3)$$

The contribution that this gives to the matrix element (3.1) is then

$$\delta(\mathbf{p}_2 - \mathbf{q}_2) (\not{p}_2 - m)_{\gamma}^{\beta} \frac{1}{2p_2^0} e^{ip_2^0 t} \Big|_{p_2^0 = \sqrt{\mathbf{p}_2^2 + m^2}} \delta(\mathbf{p}_1 - \mathbf{q}_1) \int_{\sqrt{\mathbf{p}_1^2 + (m + \mu)^2}}^{\infty} dp_1^0 e^{ip_1^0 t} \frac{\pi e^2}{2p_1^2 (-\not{p}_1 + m)} \sqrt{p_1^2 - (\mu + m)^2} \sqrt{p_1^2 - (\mu - m)^2} \left[\left(-\not{p}_1 + 3m - \frac{(p_1^2 - m^2 - 2\mu^2)(p_1^2 + \mu^2 - m^2)}{2\mu^2 p_1^2} \not{p}_1 \right) (\not{p}_1 - m)^{-1} \right]_{\alpha}^{\delta} \quad (3.4)$$

Over the range considered, i.e. $\sqrt{\mathbf{p}_1^2 + (m + \mu)^2}$ to infinity, the integrand is a continuous, smooth function, vanishing at the limits. Hence the Fourier transform is a function which vanishes at the limits. The way to see this is that the Fourier transform is a reduction of a function into harmonic waves, and so if the function is smooth, then there will be a limit to the frequency of the waves used in its composition. Thus the F.T. vanishes as $t \rightarrow \infty$. If the F.T. is not to vanish then there must be infinite frequency parts, i.e. singularities. Therefore the graph just gives a “transient”—a contribution which disappears as t becomes large. This will have no effect on the calculation of scattering amplitudes.

If we imagine that the matrix elements are integrated with suitably smooth wave functions in both the initial and final states, then we find that all of the associated graphs lead to transients, except for the ones

with a singularity built in (that is, after Fourier transforming with respect to the p^0 's and q^0 's). This rules out graphs (d)–(g) and (j)–(m): the delta functions are all integrated out, and the poles cannot be reached, since the momenta in the proliferators have p^2 either greater than $(m + \mu)^2$ or less than $(m - \mu)^2$, whereas the pole is at $p^2 = m^2$. On the other hand, in graphs (h), (i), (n) and (o) the external proliferator *can* become infinite—although not the internal one. This gives a pole, and hence a contribution which survives as $t \rightarrow \infty$.

The contribution to the matrix element from graph (h) is

$$e^2 \frac{1}{2q_1^0} \frac{1}{2p_1^0} [(\not{q}_1 + m) \gamma^a (\not{p}_1 + m)]_\alpha^\delta \frac{1}{(p_1 - q_1)^2 - \mu^2} [(\not{p}_2 - m) \gamma_a (\not{q}_2 - m)]_\gamma^\beta \frac{1}{2q_2^0} \frac{1}{p_2^2 - m^2} e^{-i(q_1^0 + q_2^0)t} \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{p}_1 - \mathbf{p}_2) \Big|_{\substack{p_1, q_1, q_2 \text{ onshell} \\ p_2^0 = q_1^0 + q_2^0 - p_1^0}} \quad (3.5)$$

We can write

$$\frac{1}{p_2^2 - m^2} = \frac{1}{(p_2^0 - E(\mathbf{p}_2))(p_2^0 + E(\mathbf{p}_2))} = \frac{1}{E(\mathbf{q}_1) + E(\mathbf{q}_2) - E(\mathbf{p}_1) - E(\mathbf{p}_2)} \frac{1}{(p_2^0 + E(\mathbf{p}_2))} \quad (3.6)$$

If the pole is the only relevant part of the amplitude, then we can replace the rest of the expression with the value of the residue at the pole. Thus $p_2^0 + E(\mathbf{p}_2) \rightarrow 2E(\mathbf{p}_2)$ (where $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$ throughout), and the contribution is

$$e^2 \frac{1}{2q_1^0 2q_2^0 2p_1^0 2p_2^0} \frac{1}{(q_1^0 + q_2^0 - p_1^0 - p_2^0)} [(\not{q}_1 + m) \gamma^a (\not{p}_1 + m)]_\alpha^\delta \frac{1}{(p_1 - q_1)^2 - \mu^2} [(\not{p}_2 - m) \gamma_a (\not{q}_2 - m)]_\gamma^\beta e^{-i(q_1^0 + q_2^0)t} \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{p}_1 - \mathbf{p}_2) \quad (3.7)$$

where *all* the four momenta are on shell now. The other pole graphs (i), (n) and (o) can be dealt with in the same way, giving us

$$\frac{e^2}{2q_1^0 2q_2^0 2p_1^0 2p_2^0} \left([(\not{q}_1 + m) \gamma^a (\not{p}_1 + m)]_\alpha^\delta \frac{1}{(p_1 - q_1)^2 - \mu^2} [(\not{p}_2 - m) \gamma_a (\not{q}_2 - m)]_\gamma^\beta + [(\not{q}_1 + m) \gamma^a (\not{q}_2 - m)]_\alpha^\beta \frac{1}{(p_1 + p_2)^2 - \mu^2} [(\not{p}_2 - m) \gamma_a (\not{p}_1 + m)]_\gamma^\delta \right) \frac{e^{-i(q_1^0 + q_2^0)t} - e^{-i(p_1^0 + p_2^0)t}}{q_1^0 + q_2^0 - p_1^0 - p_2^0} \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{p}_1 - \mathbf{p}_2) \quad (3.8)$$

All the external particles can be treated as free in this resonance approximation, but the Dirac tensor representation is not convenient for seeing the correspondence with quantum mechanics, wherein spin is defined with respect to an absolute angular momentum basis, so we introduce the linear transformations $u_{s\alpha}(p)$ and $v_{s\alpha}(p)$ having the properties

$$\begin{aligned} (\not{p} - m) u_s(p) &= 0 \\ (\not{p} + m) v_s(p) &= 0 \quad s = 1, 2 \\ \bar{u}_s(p) u_{s'}(p) &= \delta_{ss'}; \quad \bar{v}_s(p) v_{s'}(p) = -\delta_{ss'}; \quad \bar{v}u = \bar{u}v = 0; \\ u_s^+(p) u_{s'}(p) &= \frac{E(\mathbf{p})}{m} \delta_{ss'}; \quad v_s^+(p) v_{s'}(p) = \frac{E(\mathbf{p})}{m} \delta_{ss'}; \\ \bar{u}_s^\beta(p) u_{s\alpha}(p) &= \frac{1}{2m} (\not{p} + m)_\alpha^\beta; \quad \bar{v}_s^\beta(p) v_{s\alpha}(p) = \frac{1}{2m} (\not{p} - m)_\alpha^\beta \end{aligned} \quad (3.9)$$

u_s and v_s are then solutions of the Dirac equation representing respectively electrons and positrons. In the non-relativistic limit, the particles have spins aligned up and down with respect to the z -axis for each of the

values of s . Contracting these spinors appropriately with (3.8), and adding the zeroth-order contribution, we get

$$\begin{aligned} & \frac{m^2}{p_1^0 p_2^0} \delta_{r_1 s_1} \delta(\mathbf{p}_1 - \mathbf{q}_1) \delta_{r_2 s_2} \delta(\mathbf{p}_2 - \mathbf{q}_2) e^{-i(p_1^0 + p_2^0)t} \\ & + \frac{e^2 m^4}{q_1^0 q_2^0 p_1^0 p_2^0} \left(\bar{u}_{s_1}(q_1) \gamma^a u_{r_1}(p_1) \frac{1}{(p_1 - q_1)^2 - \mu^2} \bar{v}_{r_2}(p_2) \gamma_a v_{s_2}(q_2) \right. \\ & \quad \left. + \bar{u}_{s_1}(q_1) \gamma^a v_{s_2}(q_2) \frac{1}{(p_1 + p_2)^2 - \mu^2} \bar{v}_{r_2}(p_2) \gamma_a u_{r_1}(p_1) \right) \\ & \quad \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{p}_1 - \mathbf{p}_2) \frac{e^{-i(q_1^0 + q_2^0)t} - e^{-i(p_1^0 + p_2^0)t}}{q_1^0 + q_2^0 - p_1^0 - p_2^0} \end{aligned} \quad (3.10)$$

This is to be compared with

$$\begin{aligned} \langle \mathbf{q}_1, s_1; \mathbf{q}_2, s_2 | e^{iHt} | \mathbf{p}_2, r_2; \mathbf{p}_1, r_1 \rangle & = e^{i(p_1^0 + p_2^0)t} \delta_{r_1 s_1} \delta(\mathbf{q}_1 - \mathbf{p}_1) \delta_{r_2 s_2} \delta(\mathbf{q}_2 - \mathbf{p}_2) \\ & + \langle \mathbf{q}_1, s_1; \mathbf{q}_2, s_2 | V | \mathbf{p}_2, r_2; \mathbf{p}_1, r_1 \rangle \frac{e^{i(q_1^0 + q_2^0)t} - e^{i(p_1^0 + p_2^0)t}}{q_1^0 + q_2^0 - p_1^0 - p_2^0} + \mathcal{O}(V^2) \end{aligned} \quad (3.11)$$

Which we have in the purely quantum mechanical interpretation of the same system. Evidently, within the resonance approximation, up to this order, the interaction picture does exist, since a direct comparison shows that the expressions are the same (apart from a differing sign of t , which is irrelevant). If we define

$$|\mathbf{p}_2, r_2; \mathbf{p}_1, r_1\rangle = \sqrt{\frac{p_1^0 p_2^0}{m^2}} \bar{v}_{r_2}(p_2) \psi(0; \mathbf{p}_2) \bar{\psi}(0; \mathbf{p}_1) u_{r_1}(p_1) |0\rangle \quad (3.12)$$

then (3.10) takes the same form as (3.11), with

$$\begin{aligned} \langle \mathbf{q}_1, s_1; \mathbf{q}_2, s_2 | V | \mathbf{p}_2, r_2; \mathbf{p}_1, r_1 \rangle & = \frac{e^2 m^2}{\sqrt{q_1^0 q_2^0 p_1^0 p_2^0}} \left(\bar{u}_{s_1}(q_1) \gamma^a u_{r_1}(p_1) \frac{1}{(p_1 - q_1)^2 - \mu^2} \right. \\ & \quad \left. \bar{v}_{r_2}(p_2) \gamma_a v_{s_2}(q_2) + \bar{u}_{s_1}(q_1) \gamma^a v_{s_2}(q_2) \frac{1}{(p_1 + p_2)^2 - \mu^2} \bar{v}_{r_2}(p_2) \gamma_a u_{r_1}(p_1) \right) \\ & \quad \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{p}_1 - \mathbf{p}_2) \end{aligned} \quad (3.13)$$

The differential cross section for scattering of beams of different particles of momenta \mathbf{p}_1 and \mathbf{p}_2 and spin polarisations r_1 and r_2 , into momentum space regions $d^3\mathbf{q}_1$ and $d^3\mathbf{q}_2$, with spin polarisations s_1 and s_2 is given by ordinary quantum mechanics as

$$d\sigma = \frac{1}{v} d^3\mathbf{q}_1 d^3\mathbf{q}_2 |\langle \mathbf{q}_1, s_1; \mathbf{q}_2, s_2 | V | \mathbf{p}_2, r_2; \mathbf{p}_1, r_1 \rangle|^2 (2\pi)^4 \delta(q_1 + q_2 - p_1 - p_2) \quad (3.14)$$

where v is the velocity of one particle beam in a frame where the other is stationary, and the ‘‘double bar’’ matrix element is one where the three momentum conservation delta function has been extracted.

Hence

$$\begin{aligned} d\sigma & = \frac{1}{v} \frac{m}{q_1^0} d^3\mathbf{q}_1 \frac{m}{q_2^0} d^3\mathbf{q}_2 \frac{m}{p_1^0} \frac{m}{p_2^0} \\ & \quad e^4 \left| \bar{u}_{s_1}(q_1) \gamma^a u_{r_1}(p_1) \frac{1}{(p_1 - q_1)^2 - \mu^2} \bar{v}_{r_2}(p_2) \gamma_a v_{s_2}(q_2) + \right. \\ & \quad \left. \bar{u}_{s_1}(q_1) \gamma^a v_{s_2}(q_2) \frac{1}{(p_1 + p_2)^2 - \mu^2} \bar{v}_{r_2}(p_2) \gamma_a u_{r_1}(p_1) \right|^2 \\ & \quad (2\pi)^4 \delta(q_1 + q_2 - p_1 - p_2) \end{aligned} \quad (3.16)$$

The normalisations have been chosen so as to avoid factors of $(2\pi)^3$ appearing whenever possible; this means that we need to replace

$$\psi \rightarrow (2\pi)^{\frac{3}{2}}\psi; \quad A \rightarrow (2\pi)^{\frac{3}{2}}A; \quad e \rightarrow (2\pi)^{-\frac{3}{2}}e \quad (3.17)$$

to compare with the usual formulation. This gives

$$d\sigma = \frac{1}{v} \frac{m}{p_1^0} \frac{m}{p_2^0} \frac{d^3\mathbf{q}_1}{(2\pi)^3} \frac{m}{q_1^0} \frac{d^3\mathbf{q}_2}{(2\pi)^3} \frac{m}{q_2^0} e^4 \left| \bar{u}_{s_1}(q_1) \gamma^a u_{r_1}(p_1) \frac{1}{(p_1 - q_1)^2 - \mu^2} \bar{v}_{r_2}(p_2) \gamma_a v_{s_2}(q_2) + \bar{u}_{s_1}(q_1) \gamma^a v_{s_2}(q_2) \frac{1}{(p_1 + p_2)^2 - \mu^2} \bar{v}_{r_2}(p_2) \gamma_a u_{r_1}(p_1) \right|^2 (2\pi)^4 \delta(q_1 + q_2 - p_1 - p_2) \quad (3.18)$$

which is exactly what we obtain from the consideration of the Feynman graphs of fig. 6.

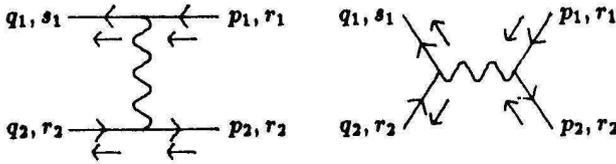


Fig. 6. Order e^2 Feynman graphs for e^+e^- elastic scattering

3.2. $e^+e^- \rightarrow \gamma\gamma$

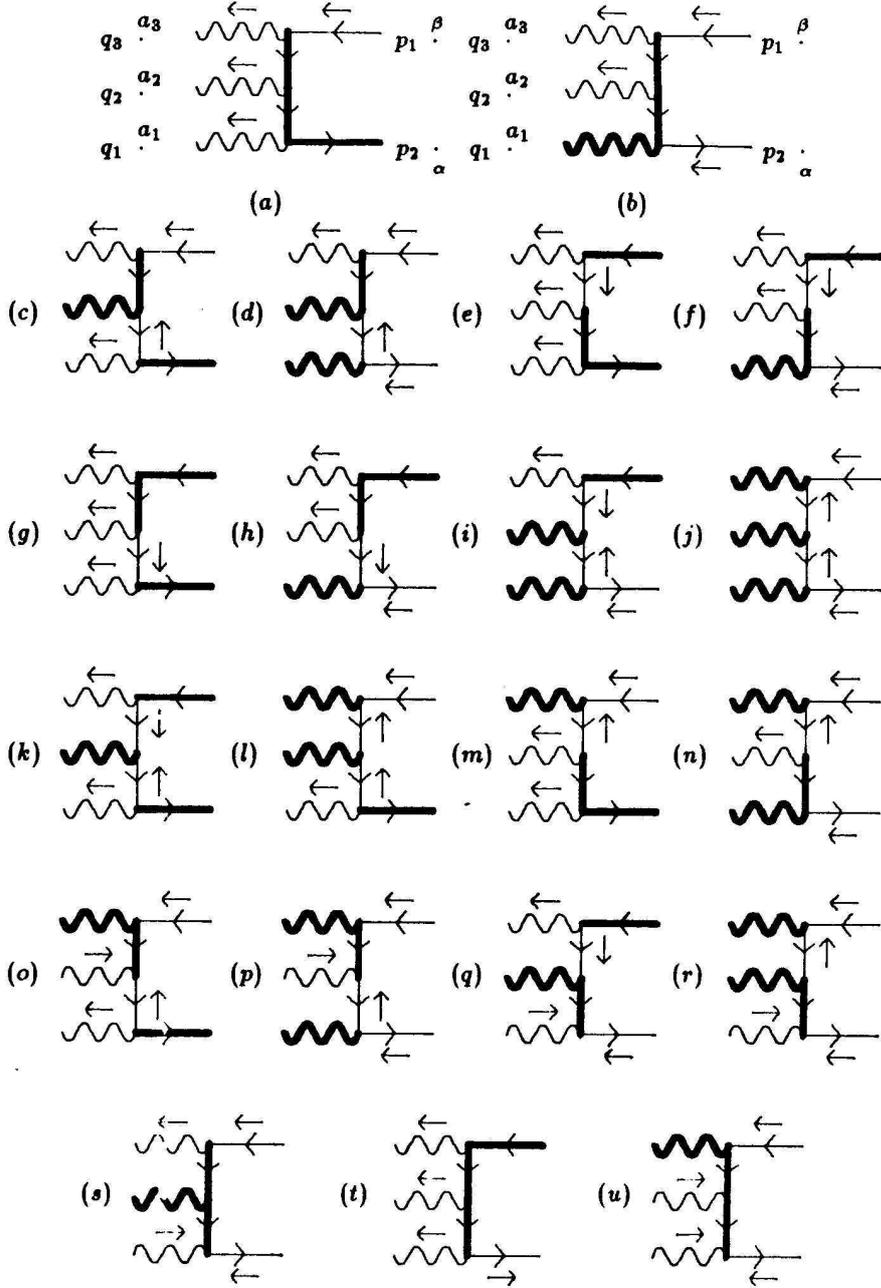


Fig. 7. Lowest order contributions to $e^+e^- \rightarrow \gamma\gamma\gamma$

The lowest order contributions to this process are listed in fig. 7. For each one of these graphs there are also five others obtained by the exchange of photon line ends. In fact certain graphs do not actually contribute to this process. Each proliferator tree must have incoming particles before it and outgoing ones after it, so (e.g.) graph (p) is a contribution to $\gamma e^+e^- \rightarrow \gamma\gamma$ but not $e^+e^- \rightarrow \gamma\gamma\gamma$. However, it may contribute as the required annihilation graph in one of the photon-permuted processes. For this particular permutation, though, it is graphs (o)–(s) [$\gamma e^+e^- \rightarrow \gamma\gamma$ only] and (t),(u) [no energetically allowed process] that we delete. Studying the remaining graphs, we see that (a)–(f) are the only ones with resonating proliferators. The graphs (c)–(f) are however rather awkward since they require an external photon which has $q^2 \leq 0$ or an external electron with $q^2 \geq (m + \mu)^2$. We cannot make sense of this: such contributions give a matrix element which does not

permit itself to be compared with normal quantum mechanics. Also, measuring instruments will measure particles which are for practical purposes “on-shell”, i.e. behaving almost as though free, with the given particle mass, so we need to discard these. This leaves us just two graphs which contribute (or rather twelve, if we include all the photon permutations), which are (a) and (b). The expressions are

$$\begin{aligned}
& e^3 \theta(q_1^0) \delta(q_1^2 - \mu^2) \left(-\eta^{a_1 b_1} + \frac{q_1^{a_1} q_1^{b_1}}{\mu^2} \right) \theta(q_2^0) \delta(q_2^2 - \mu^2) \left(-\eta^{a_2 b_2} + \frac{q_2^{a_2} q_2^{b_2}}{\mu^2} \right) \\
& \quad \theta(q_3^0) \delta(q_3^2 - \mu^2) \left(-\eta^{a_3 b_3} + \frac{q_3^{a_3} q_3^{b_3}}{\mu^2} \right) (-1) \\
& \quad \left[(\not{p}_2 + m)^{-1} \gamma_{b_1} (\not{p}_2 - \not{q}_1 + m)^{-1} \gamma_{b_2} (\not{q}_3 - \not{p}_1 + m)^{-1} \gamma_{b_3} \theta(p_1^0) (\not{p}_1 + m) \delta(p_1^2 - m^2) \right]_{\alpha}^{\beta} \\
& \quad \delta(q_1 + q_2 + q_3 - p_1 - p_2) \\
& + e^3 \theta(q_3^0) \delta(q_3^2 - \mu^2) \left(-\eta^{a_3 b_3} + \frac{q_3^{a_3} q_3^{b_3}}{\mu^2} \right) \theta(q_2^0) \delta(q_2^2 - \mu^2) \left(-\eta^{a_2 b_2} + \frac{q_2^{a_2} q_2^{b_2}}{\mu^2} \right) \\
& \quad (q_1^2 - \mu^2)^{-1} \left(\eta^{a_1 b_1} - \frac{q_1^{a_1} q_1^{b_1}}{\mu^2} \right) [(\not{p}_2 - m) \gamma_{b_1} \theta(p_2^0) \delta(p_2^2 - m^2) (\not{q}_1 - \not{p}_2 - m)^{-1} \gamma_{b_2} \\
& \quad (\not{p}_1 - \not{q}_3 - m)^{-1} \gamma_{b_3} \theta(p_1^0) \delta(p_1^2 - m^2) (\not{p}_1 + m)]_{\alpha}^{\beta} \delta(q_1 + q_2 + q_3 - p_1 - p_2) \quad (3.19)
\end{aligned}$$

We have used

$$-\frac{\left(\eta^{ab} - \frac{q^a q^b}{\mu^2} \right)}{q^2 - \mu^2} \quad \text{rather than} \quad -\frac{\eta^{ab}}{q^2 - \mu^2}$$

for the photon proliferator, which makes no difference since current is conserved at each vertex. Also, there is an additional factor of (-1) for the second graph. This arises because, strictly, the second graph has not been drawn correctly: the charge flow in the outgoing tree should be *upwards*. This means that the fermion lines will cross when attaching to this, giving a factor of (-1). The contribution to

$$\langle 0 | A^{a_3}(t; -\mathbf{q}_3) A^{a_2}(t; -\mathbf{q}_2) A^{a_1}(t; -\mathbf{q}_1) \psi_{\alpha}(0; \mathbf{p}_2) \bar{\psi}^{\beta}(0; -\mathbf{p}_1) | 0 \rangle \quad (3.20)$$

is thus

$$\begin{aligned}
& e^3 \frac{1}{2q_1^0 2q_2^0 2q_3^0 2p_1^0 2p_2^0} \left(-\eta^{a_1 b_1} + \frac{q_1^{a_1} q_1^{b_1}}{\mu^2} \right) \left(-\eta^{a_2 b_2} + \frac{q_2^{a_2} q_2^{b_2}}{\mu^2} \right) \left(-\eta^{a_3 b_3} + \frac{q_3^{a_3} q_3^{b_3}}{\mu^2} \right) \\
& \quad [(\not{p}_2 - m) \gamma_{b_1} (\not{q}_1 - \not{p}_2 - m)^{-1} \gamma_{b_2} (\not{p}_1 - \not{q}_3 - m)^{-1} \gamma_{b_3} (\not{p}_1 + m)]_{\alpha}^{\beta} \\
& \quad \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 - \mathbf{p}_1 - \mathbf{p}_2) \frac{e^{-i(q_1^0 + q_2^0 + q_3^0)t} - e^{-i(p_1^0 + p_2^0)t}}{q_1^0 + q_2^0 + q_3^0 - p_1^0 - p_2^0} \quad (3.21)
\end{aligned}$$

where we have also taken the residue of the pole, and all four-momenta are now on shell.

The orthonormal e^+e^- and $\gamma\gamma\gamma$ states of ordinary quantum mechanics would be related to the states generated here by

$$|\mathbf{p}_2, r_2; \mathbf{p}_1, r_1\rangle = \sqrt{\frac{p_1^0 p_2^0}{m^2}} \bar{v}_{r_2}(p_2) \psi(0; \mathbf{p}_2) \bar{\psi}(0; \mathbf{p}_1) u_{r_1}(p_1) | 0 \rangle$$

and

$$|\mathbf{q}_1, i_1; \mathbf{q}_2, i_2; \mathbf{q}_3, i_3\rangle = \frac{1}{\sqrt{3!}} \sqrt{2q_1^0} \epsilon^{(i_1)}(q_1) \cdot A(q_1) \sqrt{2q_2^0} \epsilon^{(i_2)}(q_2) \cdot A(q_2) \sqrt{2q_3^0} \epsilon^{(i_3)}(q_3) \cdot A(q_3) | 0 \rangle \quad (3.22)$$

where $\epsilon^{(i)}(q)$ are spin polarisation tensors, satisfying

$$q \cdot \epsilon^{(i)}(q) = 0; \quad \epsilon^{(i)*}(q) \cdot \epsilon^{(j)}(q) = \delta^{ij}; \quad \sum_i \epsilon_a^{(i)*}(q) \epsilon_b^{(i)}(q) = \left(-\eta_{ab} + \frac{q_a q_b}{\mu^2} \right) \quad (3.23)$$

This gives us a quantum-mechanics-style reduced matrix element for the process thus:

$$\langle \mathbf{q}_1, i_1; \mathbf{q}_2, i_2; \mathbf{q}_3, i_3 \| V \| \mathbf{p}_2, r_2; \mathbf{p}_1, r_1 \rangle = \frac{e^3}{\sqrt{2q_1^0 2q_2^0 2q_3^0}} \sqrt{\frac{m^2}{p_1^0 p_2^0}} \frac{1}{\sqrt{3!}} (\epsilon^{(i_1)*b_1}(q_1) \epsilon^{(i_2)*b_2}(q_2) \epsilon^{(i_3)*b_3}(q_3) + 5 \text{ exchanges}) \bar{v}_{r_2}(p_2) \gamma_{b_1} (\not{q}_1 - \not{p}_2 - m)^{-1} \gamma_{b_2} (\not{p}_1 - \not{q}_3 - m)^{-1} \gamma_{b_3} u_{r_1}(p_1) \quad (3.24)$$

and hence a differential cross-section of

$$d\sigma = \frac{1}{v} \frac{d^3 \mathbf{q}_1}{2q_1^0} \frac{d^3 \mathbf{q}_2}{2q_2^0} \frac{d^3 \mathbf{q}_3}{2q_3^0} \frac{m}{p_1^0} \frac{m}{p_2^0} \frac{1}{3!} \left| e^3 (\epsilon^{(i_1)*b_1}(q_1) \epsilon^{(i_2)*b_2}(q_2) \epsilon^{(i_3)*b_3}(q_3) + 5 \text{ exchanges}) \bar{v}_{r_2}(p_2) \gamma_{b_1} (\not{q}_1 - \not{p}_2 - m)^{-1} \gamma_{b_2} (\not{p}_1 - \not{q}_3 - m)^{-1} \gamma_{b_3} u_{r_1}(p_1) \right|^2 (2\pi)^4 \delta(q_1 + q_2 + q_3 - p_1 - p_2) \quad (3.25)$$

which upon making the necessary rescaling $e \rightarrow (2\pi)^{-\frac{3}{2}} e$ will be seen to be identical to the result obtained from the consideration of the Feynman graphs of fig. 8.

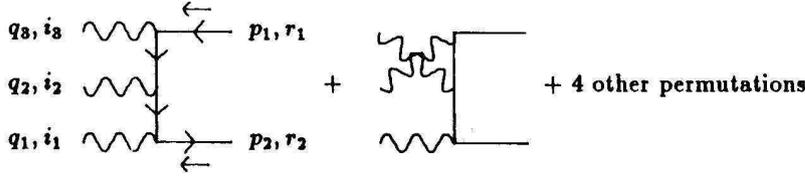


Fig. 8. Lowest order Feynman graphs for $e^+ e^- \rightarrow \gamma \gamma \gamma$

3.3. THE GENERAL CASE

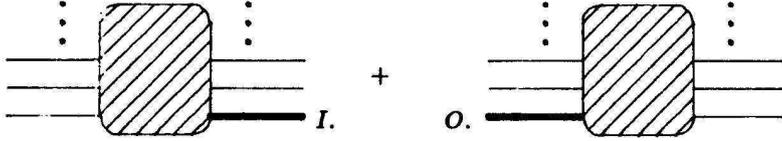


Fig. 9. The resonance graphs for a general scattering process

For any scattering process, the lowest-order contributions will always be “tree” graphs, and there will be a pair of graphs of the form given in fig. 9: i.e. a single external proliferator, which is at the bottom of the stacks of respectively incoming and outgoing lines. The lines here are generic (i.e. γ , e^+ or e^- particles), and provided that the process is energetically allowed, these will give resonance contributions arising from this external proliferator. The internal network necessarily comprises solely of proliferators. The propagators are all integrated out with respect to their time components, which gives

$$\frac{1}{2q_0} \left(-\eta_{aa'} + \frac{q_a q_{a'}}{\mu^2} \right), \quad \frac{1}{2q_0} (\not{q} + m) \quad \text{and} \quad \frac{1}{2q_0} (\not{q} - m) \quad (3.26)$$

for the cases of photons, electrons and positrons respectively (where all momenta are on shell). The integration of the time component of the proliferator momentum is done by the time-component part of the four-momentum delta function. This gives an energy conservation resonance pole. Taking the residue of this in the rest of the expression, we have

$$\begin{aligned} (\mp) \frac{1}{2q_0} \left(-\eta_{aa'} + \frac{q_a q_{a'}}{\mu^2} \right) \frac{1}{E_f - E_i}, \quad (\mp) \frac{1}{2q_0} (\not{q} + m) \frac{1}{E_f - E_i} \\ \text{and} \quad (\pm) \frac{1}{2q_0} (\not{q} - m) \frac{1}{E_f - E_i} \end{aligned} \quad (3.27)$$

for photons, electrons and positrons. The top sign is for appearance in the final state, the bottom for appearance in the initial state. In the resonance approximation, the value of the internal expression is the same for each graph, apart from a possible sign. This arises because charge in a proliferator tree attached to an incoming line flows downwards, whereas in an outgoing tree it flows upwards, and in going from one to the other we may cause fermion lines to cross or uncross, each one generating a factor of (-1). A rather involved argument shows us that the difference between the two internal expressions is $(-1)^{e_I^+ + e_O^+}$, where e_I^+ is unity if the proliferator I is a positron and zero otherwise. Similarly for e_O^+ . This is because of fermion line crossings. Hence the value of the resonance parts of the two graphs is

$$(-)^{e_I^+} \frac{e^{-iE_f t} - e^{-iE_i t}}{E_f - E_i} (\text{outgoing factors}) (\text{internal expression}) (\text{incoming factors}) \quad (3.28)$$

(the ‘‘internal expression’’ applying to the graph with the incoming proliferator). The factors applied to the internal expression are of the form of propagators, i.e. those of (3.26) which includes the place where the proliferator was before.

To relate this matrix element to one of ordinary quantum mechanics, we need to multiply incoming lines by

$$\sqrt{2q_0} \epsilon_a^{(i)}(q), \quad \sqrt{\frac{q_0}{m}} u_r(q) \quad \text{and} \quad \sqrt{\frac{q_0}{m}} \bar{v}_r(q) \quad (3.29)$$

for respectively photons, electrons and positrons. The Pauli adjoints of these are applied to outgoing lines. Contracting these with the factors obtained before, we then have

$$-\frac{1}{\sqrt{2q_0}} \epsilon_a^{(i)}(q), \quad \sqrt{\frac{m}{q_0}} u_r(q) \quad \text{and} \quad -\sqrt{\frac{m}{q_0}} \bar{v}_r(q) \quad (3.30)$$

as the factors to be applied to the internal expression, the adjoints being used for outgoing lines. There is also a factor of $1/\sqrt{m!}$ for each m -repeated incoming or outgoing particle.

If we include the external proliferator, then we can imagine the graph as built up with the rules

$$\begin{aligned} \frac{i}{\not{q} - m} & \quad \text{for internal fermion lines, where } q \text{ is the} \\ & \quad \text{momentum flowing parallel to the charge line} \\ \frac{-i\eta^{ab}}{q^2 - \mu^2} & \quad \text{for internal photon lines} \\ -ie\gamma_a & \quad \text{for each vertex} \end{aligned}$$

which are those of ordinary Feynman graph analysis. This follows because the number of vertices is the same as the number of proliferators, so the i discrepancy in the lines multiplies the $-i$ discrepancy in the vertices to give no overall change. However the external proliferator is put on shell so an extra factor of i is spare. The matrix element is thus

$$\frac{e^{-iE_f t} - e^{-iE_i t}}{E_f - E_i} i(-)^{n_\gamma + n_{e^+}} \sqrt{S} \left(\frac{m}{q_0} \right)^{(n_{e^+} + n_{e^-})/2} \left(\frac{1}{2q_0} \right)^{n_\gamma/2} [\text{Feynman graph amplitude}] \quad (3.31)$$

where $n_{e_{\pm}}$ are the total number of external e_{\pm} lines and n_{γ} is the total number of external photon lines.

$$S = \prod_i \frac{1}{m_i!}$$

is the factor for repeated particles of the same type. The $(-)^{e_i^{\dagger}}$ belongs to the Feynman amplitude: if we look closely at the S -matrix reduction, we see that the signs from fermion crossings are the same except that the incoming proliferator is not treated any differently to the other lines and so the charge line which is generated will have a loop in it if it is a positron (to make it flow the right way into the vertex), and hence a sign.

We may now sum all of the amplitudes if more than one graph contributes to the process. We may also make the rescaling $e \rightarrow (2\pi)^{-\frac{3}{2}}e$. The number of vertices is two greater than the number of external lines (since it is built of ‘tree’ diagrams). Also, there are two incoming lines, so the effect of the rescaling is to put factors of $(2\pi)^{-3}$ on each outgoing line. Thus the cross-section for the process is given by

$$d\sigma = \frac{1}{v} (\text{incoming factors}) |\mathcal{M}|^2 (\text{outgoing factors}) (2\pi)^4 \delta(p_1 + p_2 - \sum_{i=1}^n q_i) S \quad (3.32)$$

where the incoming factors are $1/2p_0$ for a photon and m/p_0 for a fermion. The outgoing factors are $d^3\mathbf{q}/((2\pi)^3 2q_0)$ for photons and $d^3\mathbf{q}/(2\pi)^3(m/q_0)$ for fermions. \mathcal{M} is the sum of Feynman graphs and S is the statistical factor

$$S = \prod_i \frac{1}{m_i!}$$

for each m_i -repeated particle type, but this time only over the *final* state particles. The factor $\frac{1}{2}$ from possible identical particles in the initial state is cancelled by a factor of two which appears in the quantum mechanical cross-section formula for this case.

What we have written down, of course, is none other than the cross-section arising from the tree Feynman graphs for the process in question.

4. Conclusion and outlook

In formulating any physical theory it is necessary to start by eliminating that which is *definitely* wrong, and then to find what could be right from whatever is left. This process is painful in the case of relativistic quantum field theory as, at present, *nothing* is left. A theory which is plagued with infinities is *definitely* wrong, as is a theory which violates Haag’s theorem. The author nurtured the hope, at least initially, that curing one would also cure the other, but this was clearly misguided. This is necessarily so, as we are simply not allowed to expand in the coupling constant. To see this, try expanding a wave function of the Hydrogen atom out in powers of e and see how much sense that makes.

So, to conclude, we have a technique which at best solves some important problems of quantum field theory, and provides a pointer to a possible future theory which is completely free from infinities. At worst it simply reminds us that, over sixty years after it was first thought of, we still have no interacting relativistic quantum field theory.

References

- [1] Hall, D. and Wightman, A., Mat. Fys. Medd. Dansk. Vid. Selskapet 31 5 (1957)
See also Streater, R. and Wightman, A., PCT, spin, statistics and all that.
p. 165. Benjamin-Cummings, N.Y., 1979
- [2] Oakley, C., ‘Does the interaction picture exist?’ Hatfield Polytechnic preprint, 1986

The ‘usual formulation of Q.E.D.’ means that presented in
Bjorken, J. and Drell, S., Relativistic quantum fields, McGraw-Hill, 1965 or
Itzykson, C., and Zuber, J.-B., Quantum field theory, McGraw-Hill, 1980